Non Intrusive Reduced Basis method (NIRB)

The Two-grids method

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Introduction
The two-grids method is non intrusive

Industrial context $\rightarrow$ black box solver (BB)

Non intrusive reduced basis method useful for:

- Optimization parameters fitting
- High fidelity real-time simulations

Goal: Solve for several parameters the same parameter dependent problem and reduce the computational costs

Several methods:

- Finite Element method
- Extension to Finite Volume method

Two examples of applications

- **Offshore wind Farm**

  A way to reduce computational costs of an offshore wind farm to optimize the position of the wind turbines.

- **Spatial Variability**

  What if we could consider a non physically acceptable truncated domain?

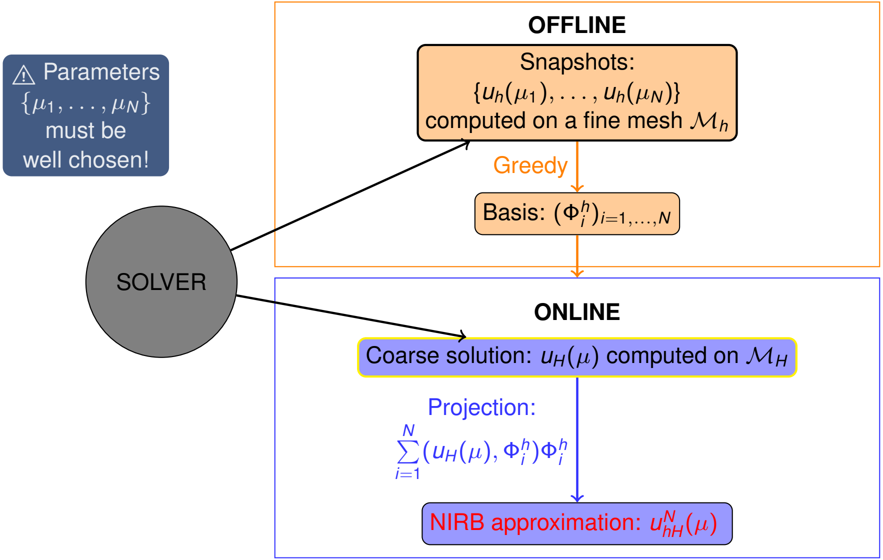
A model problem

$$\begin{cases} -\mathrm{div}(A(\mu)\nabla u) = f \text{ in } \Omega, & \text{(1a)} \\ u = 0 \text{ on } \partial\Omega, & \text{(1b)} \end{cases}$$

- $u(\mathbf{x};\mu)$: Unknowns (u_h on the fine mesh $\mathcal{T}_h$, u_H on the coarse mesh $\mathcal{T}_H$).
- $\mu \in S$: Variable parameter,

$f \in L^2(\Omega)$,
$A : S \times \Omega \to \mathcal{M}_d(\mathbb{R})$ is measurable, bounded, uniformly elliptic, and $A(\mathbf{x})$ is symmetric for a.e. $\mathbf{x} \in \Omega$.

NIRB scheme OFFLINE/ONLINE

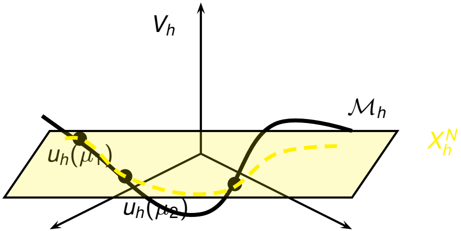


How to choose the parameters

- Greedy algorithm
- Observing the decay of eigenvalues with an SVD

Kolmogorov n-width must be small [1] $\mathcal{M}_h = \{u_h(\mu) \in V_h \mid \mu \in \mathcal{P}\}$ is a subset of a Banach space V_h.
The Kolmogorov n-width of $\mathcal{M}_h$ in V_h is

$$d_n(\mathcal{M}_h, V_h) = \inf_{Y_n}\{ \sup_{x\in\mathcal{M}_h} (\inf_{y\in Y_n} \|x-y\|_{V_h}); \mathrm{Y_n \text{ is a } n\text{-dimensional subspace of }} V_h\}. \qquad (2)$$



[1] A. Buffa, Y. Maday, A.T. Patera, C. Prudhomme, and G. Turinici, *A Priori convergence of the greedy algorithm for the parameterized reduced basis.*2010

NIRB Offline stage

1. Compute the approximations $\{u_h(\mu_i)\}_{i=1,\dots,N}$.

2. We consider:
   - A greedy algorithm with a Gram-Schmidt procedure $\rightarrow L^2$ orthonormalization.
   - Complemented by the following problem:
     Find $\Phi^h \in X_h^N$, and $\lambda \in \mathbb{R}$ such that

$$\forall v \in X_h^N, \int_\Omega \nabla \Phi^h \cdot \nabla v = \lambda \int_\Omega \Phi^h \cdot v, \tag{3}$$

$\rightarrow L^2(\Omega)$ and $H^1(\Omega)$ orthogonalization.

$X_h^N = Vect\{\Phi_1^h, \dots, \Phi_N^h\}$

NIRB Online stage

3. Solve problem on the coarse mesh $\mathcal{T}_H$ where $H >> h$ with μ.
4. $\alpha_i^H = \int_\Omega I^h(u_H(\mu)) \cdot \Phi_i^h$ and output: $u_{Hh}^N = \sum_{i=1}^{N} \alpha_i^H \Phi_i^h$.
5. (Optional) Post-Treatment (PT)

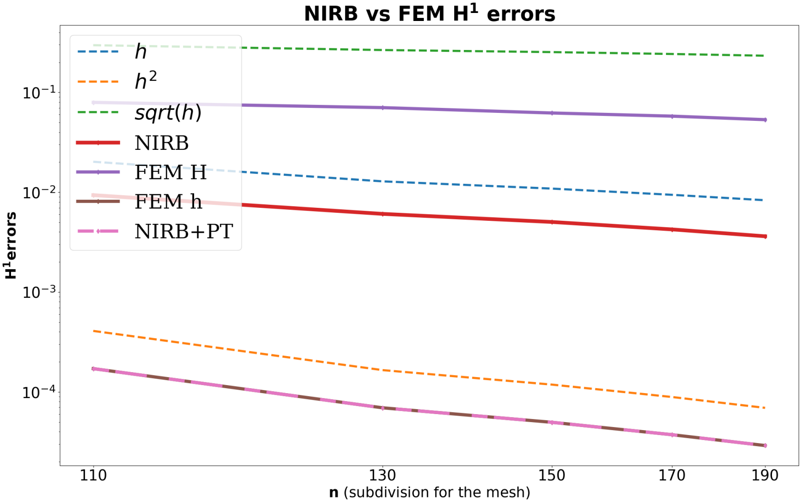
$$\left\| u(x;\mu) - \sum_{k=1}^{N} (u_H(\mu), \phi_k^h)\ \phi_k^h \right\|_{H^1} \leq \overbrace{\epsilon}^{T_1} + \underbrace{C_1 h}_{T_2} + \overbrace{C_2 H^2}^{T_3} \sim \ Ch$$

if $H^2 \sim h$

where C_1, C_2 are constants independent of h and H.[2]

[2] Rachida Chakir, Yvon Maday. *A two-grid finite-element/reduced basis scheme for the approximation of the solution of parameter dependent PDE.* 2009

Results with FE solver



Polytopal mesh for FV

$$X_{\mathcal{D},0} = \{v = ((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{F}}) :$$
$$v_K \in \mathbb{R}, v_\sigma \in \mathbb{R},$$
$$v_\sigma = 0 \text{ if } \sigma \in \mathcal{F}_{ext}\}.$$

Goal:
Extend FE estimate
to FV solver

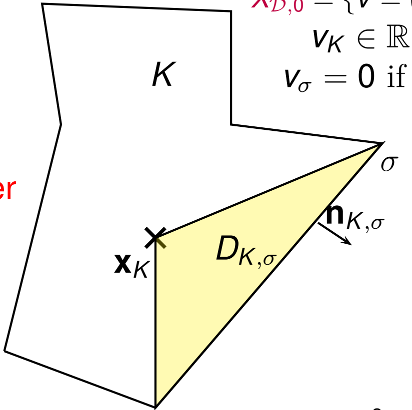


Figure: A cell K of a polytopal mesh [3]

[3] J. Droniou, R. Eymard, T. Gallouët, C. Guichard, R. Herbin. *The gradient discretisation method.* 2018

Gradient Discret scheme

$$\begin{cases} -\operatorname{div}(A(\mu)\nabla u) = f \text{ in } \Omega, & (4a) \\ u = 0 \text{ on } \partial\Omega. & (4b) \end{cases}$$

Variational Gradient Scheme [4]

Find $u_{\mathcal{D}} \in X_{\mathcal{D},0}$ such that, $\forall v_{\mathcal{D}} \in X_{\mathcal{D},0}$,

$$\int_\Omega A(\mu)\nabla_{\mathcal{D}} u_{\mathcal{D}} \cdot \nabla_{\mathcal{D}} v_{\mathcal{D}} = \int_\Omega f\, \Pi_{\mathcal{D}} v_{\mathcal{D}}. \quad (5)$$

[4] J. Droniou, R. Eymard, T. Gallouët, C. Guichard, R. Herbin. *The gradient discretisation method.* 2018

Error estimate on FV scheme

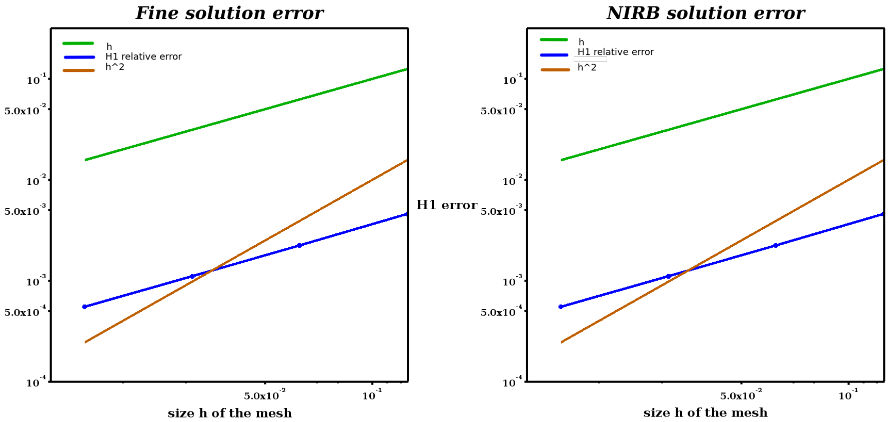
1. $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)$:
   $\Pi_{\mathcal{D}} v(\mathbf{x}) = v_K$ on K.
2. $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)^d$:
   $\nabla_{\mathcal{D}} v(\mathbf{x}) = \nabla_K v + S$ on $D_{K,\sigma}$, where S ensures stability and
   $\nabla_K v = \frac{1}{|K|} \sum_{\sigma \in \mathcal{F}_K} |\sigma| v_\sigma \mathbf{n}_{K,\sigma}$.

A norm on $X_{\mathcal{D},0}$: $\|\cdot\|_{\mathcal{D}} = \|\nabla_{\mathcal{D}} \cdot\|_{L^2(\Omega)^2}$.

H^1 error estimate

Main result: $\left\| u(\mu) - u_{Hh}^N(\mu) \right\|_{\mathcal{D}} \leq Ch$, if $H^2 \sim h$.

Results with FV solver



NIRB application: 2D Wind turbine

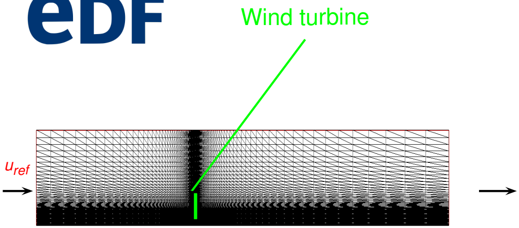


Figure 1: Mesh for one wind turbine

u_{ref}: Variable parameter

- 2D mesh with 6500 cells, thinner around the wind turbine.
- Characteristic length D: 126m, corresponds to the rotor diameter.
- Hub height: 95.6m.
- Wind turbine rotor is represented in the movement equation by adding a source term.
- Boundary Condition: u_{ref} at the inlet.
- Initial Condition: u_{ref} set in the domain.

Results for the application

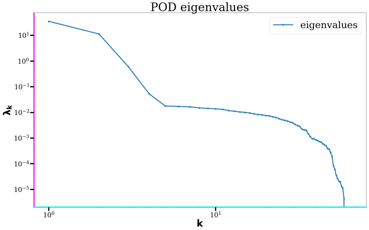


Figure 2: Decrease of the eigenvalues of the POD

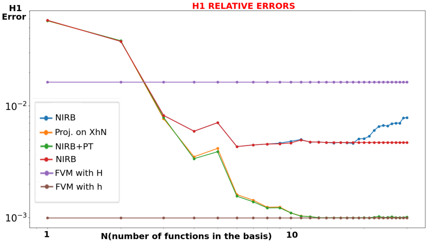
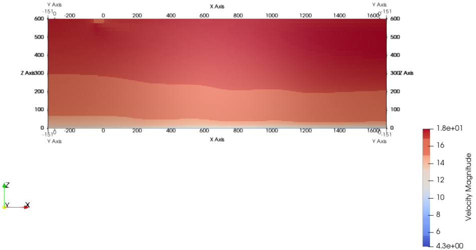
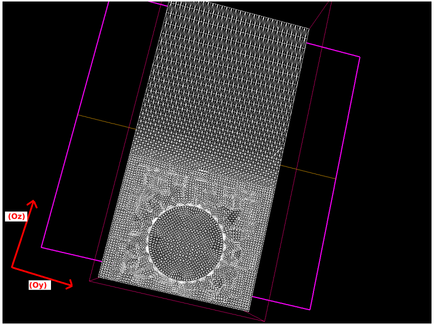


Figure 3: H^1 errors of the velocity on the region of interest

Wind turbines in 3D

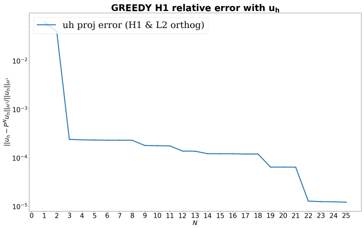


Wind canal

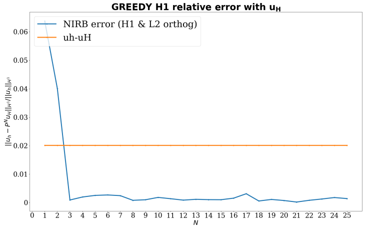


One wind turbine mesh ($\mathcal{N} \sim 500\ 000$)

Results for 3D application



Projection H^1 error



NIRB H^1 error

Wind turbine n°3 approximation

NIRB

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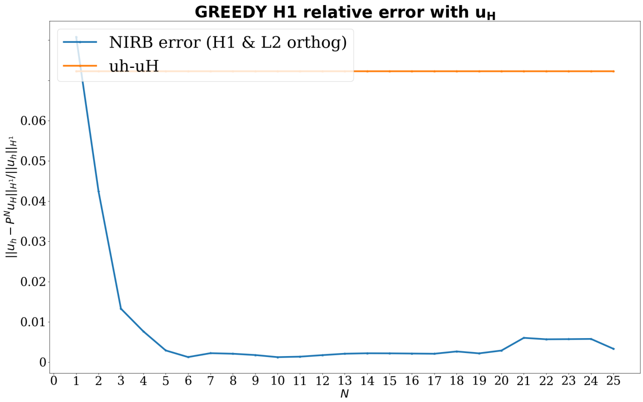
NIRB method
- Offline
- Online
- Results with FE solver

Finite volume solver
- Error estimate
- Results with FV solver

Applications
- **EDF applications**
- Truncated domain

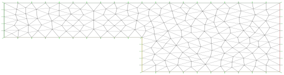
Conclusions and perspectives

We may approximate the third wind turbine with the second one!



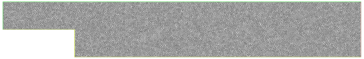
NIRB H^1 error

Application on a truncated domain



Greedy on coarse mesh $(\Phi_i^H)_{i=1,\cdots,N_1}$

$\rightsquigarrow$



Greedy on fine mesh $(\Phi_i^h)_{i=1,\cdots,N_2}$

$$(A_i)_k = (u_H(\mu_k), \Phi_i^H)_{L^2}, \forall k = 1, \cdots, Ntrain, \tag{6}$$

$$(B_i)_k = (u_h(\mu_k), \Phi_i^h)_{L^2}, \forall k = 1, \cdots, Ntrain, \tag{7}$$

$$D = (A_1, \cdots, A_N) \in \mathbb{R}^{Ntrain \times N_1}, \tag{8}$$

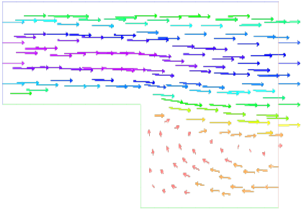
$$R_i = (D^T D + \lambda I_N)^{-1} D^T B_i, \ \forall i = 1, \cdots, N_2. \tag{9}$$

$$u_{Hh}^N(\mu) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} R_{ij} (u_H(\mu), \Phi_j^H) \Phi_i^h. \tag{10}$$

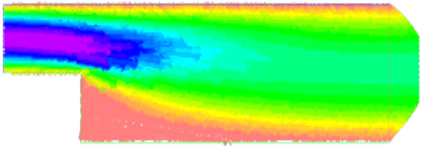
Result

size of coarse mesh = 0.14 size of fine mesh = 0.03

H1 NIRB error for Reynolds = 200: 1.0549e-05



FE solution on the coarse mesh



Nirb approximation

Conclusion and Perspectives

1. Generalization of NIRB method to FV solvers with classical error estimate.
2. Numerical results with FV solver on wind turbines in accordance with expectations in 2D and 3D.
3. Efficient NIRB method on a truncated domain.

Perspectives

- Extend 3D wind turbines to offshore wind farm,
- Use a truncated domain on wind turbines,
- Generalize to other FV schemes,
- Development of a library of non intrusive reduced basis methods in process.

Thank you for your attention!

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The rectification method

$(u_H^i, \Phi_j) \to (u_h^i, \Phi_j)$

$$(A_i)_k = (u_H(\mu_k), \Phi_i^h)_{L^2}, \forall k = 1, \cdots, Ntrain \quad (11)$$

$$(B_i)_k = (u_h(\mu_k), \Phi_i^h)_{L^2}, \forall k = 1, \cdots, Ntrain \quad (12)$$

$$D = (A_1, \cdots, A_N) \in \mathbb{R}^{Ntrain \times N} \quad (13)$$

$$T_i = (D^T D + \lambda I_N)^{-1} D^T B_i, \ \forall i = 1, \cdots, N. \quad (14)$$

$$u_{Hh}^N(\mu) = \sum_{i,j=1}^{N} T_{ij}(u_H(\mu), \Phi_j^h)\Phi_i^h \quad (15)$$