

KKT: exercise 1 (equality constraint)

Exercise

Consider the problem

$$\min_{(x,y) \in \mathbb{R}^2} f(x,y) = x + y, \quad h(x,y) = x^2 + y^2 = 0.$$

- 1 Determine the feasible set and the global minimizer.
- 2 Write the Lagrangian and the KKT conditions.
- 3 Does LICQ hold at the minimizer?
- 4 Can the minimizer satisfy the KKT conditions?

- $\bar{x} = (0, 0)$ is the unique global minimizer.

- $$L(x, y, \mu) = x + y + \mu(x^2 + y^2), \quad \mu \in \mathbb{R}.$$

$$\nabla_{(x,y)} L = (1 + 2\mu x, 1 + 2\mu y) = (0, 0).$$

- $$\nabla_{(x,y)} L(0,0,\mu) = (1,1) \neq (0,0).$$

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KKT: exercise 2 (dependent active constraints)

Exercise

Consider

$$\min_{(x,y) \in \mathbb{R}^2} f(x,y) = x$$

subject to

$$g_1(x,y) = x^2 - y \leq 0, \quad g_2(x,y) = x^2 + y \leq 0.$$

- 1 Show that the feasible set consists of a single point.
- 2 Identify the active constraints and check LICQ.
- 3 Write the Lagrangian and the KKT conditions.
- 4 Does the global minimizer satisfy KKT?

KKT: exercise 2 – solution

- 1 Feasibility gives $y \geq x^2$ and $y \leq -x^2$. Thus $2x^2 \leq 0$, so $x = 0$ and $y = 0$:

$\bar{x} = (0, 0)$ is the unique global minimizer.

- 2 Both constraints are active, with

$$\nabla g_1(0, 0) = (0, -1), \quad \nabla g_2(0, 0) = (0, 1).$$

They are nonzero but linearly dependent: **LICQ fails**.

- 3 The Lagrangian is

$$L = x + \lambda_1(x^2 - y) + \lambda_2(x^2 + y), \quad \lambda_1, \lambda_2 \geq 0.$$

- 4 At the minimizer,

$$\nabla_{(x,y)} L(0, 0, \lambda_1, \lambda_2) = (1, -\lambda_1 + \lambda_2) \neq (0, 0).$$

No KKT multipliers exist, despite global optimality.

KKT: exercise 3 (convex problem)

Exercise

Consider

$$\min_{(x,y) \in \mathbb{R}^2} (x-2)^2 + (y-1)^2$$

subject to

$$g_1(x,y) = x + y - 2 \leq 0, \quad g_2(x,y) = -x \leq 0.$$

- 1 Write the Lagrangian and KKT conditions.
- 2 Verify that $(\bar{x}, \bar{y}) = (3/2, 1/2)$ satisfies KKT and determine the multipliers.
- 3 Check LICQ at this point.
- 4 Prove that the point is the unique global minimizer.

$$L = (x - 2)^2 + (y - 1)^2 + \lambda_1(x + y - 2) - \lambda_2 x.$$
$$\begin{cases} 2(x-2) + \lambda_1 - \lambda_2 = 0, \\ 2(y-1) + \lambda_1 = 0, \\ x+y \leq 2, \ x \geq 0, \\ \lambda_1, \lambda_2 \geq 0, \\ \lambda_1(x+y-2) = 0, \ \lambda_2 x = 0. \end{cases}$$
$$\bar{\lambda}_1 = 1, \quad \bar{\lambda}_2 = 0.$$

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Sensitivity analysis: exercise 4

Exercise

A company chooses two production quantities $x, y \geq 0$. Its production cost (in k€) is

$$f(x, y) = \frac{1}{2}((x - 3)^2 + (y - 3)^2).$$

The total production capacity is $b > 0$:

$$\min_{x, y \geq 0} f(x, y), \quad x + y \leq b.$$

- 1 Write the KKT conditions.
- 2 For $b = 4$, determine the solution and its multipliers.
- 3 For $0 < b < 6$, determine the value function $V(b)$.
- 4 Verify that $V'(4) = -\bar{\lambda}$, where $\bar{\lambda}$ is the capacity-constraint multiplier.
- 5 Additional capacity costs 0.6 k€ per unit. Should the company increase capacity slightly above $b = 4$?

Sensitivity analysis: exercise 4 – solution (1)

The Lagrangian is

$$L = \frac{1}{2}((x-3)^2 + (y-3)^2) + \lambda_1(x+y-b) - \lambda_2x - \lambda_3y,$$

with $\lambda_1, \lambda_2, \lambda_3 \geq 0$. The KKT conditions are

$$\begin{cases} x - 3 + \lambda_1 - \lambda_2 = 0, \\ y - 3 + \lambda_1 - \lambda_3 = 0, \\ x + y \leq b, \quad x, y \geq 0, \\ \lambda_1(x + y - b) = 0, \quad \lambda_2x = \lambda_3y = 0. \end{cases}$$

At $b = 4$, take $x = y = 2$. The capacity constraint is active, while the nonnegativity constraints are inactive:

$$(\bar{x}, \bar{y}) = (2, 2), \quad \bar{\lambda}_1 = 1, \quad \bar{\lambda}_2 = \bar{\lambda}_3 = 0.$$

The problem is convex and the objective strictly convex: this is the **unique global minimizer**.

Sensitivity analysis: exercise 4 – solution (2)

For $0 < b < 6$, the capacity constraint is active and $x, y > 0$. Thus $\lambda_2 = \lambda_3 = 0$ and stationarity gives

$$x = y = 3 - \lambda_1, \quad x + y = b.$$

Consequently,

$$x(b) = y(b) = \frac{b}{2}, \quad \lambda_1(b) = 3 - \frac{b}{2}.$$

The optimal value is

$$V(b) = \left(3 - \frac{b}{2}\right)^2 = \frac{(6 - b)^2}{4}.$$

Hence

$$V'(b) = -\left(3 - \frac{b}{2}\right) = -\lambda(b).$$

At $b = 4$, $V'(4) = -1$: one extra unit of capacity reduces the optimal production cost by approximately 1 k€. Its price is 0.6 k€; the net marginal gain is $1 - 0.6 = 0.4$ k€.

Exercise 5: quadratic penalty versus augmented Lagrangian

Exercise

Consider

$$\min_{x \in \mathbb{R}} x, \quad h(x) = x = 0.$$

- 1 Determine the constrained solution $\bar{x}$ and its KKT multiplier $\bar{\mu}$.
- 2 Minimize $Q_c(x) = x + \frac{c}{2}x^2$ for $c > 0$. Does a finite c give $\bar{x}$?
- 3 Minimize $L_c(x, \mu) = x + \mu x + \frac{c}{2}x^2$. What happens when $\mu = \bar{\mu}$?
- 4 Starting from any μ_0 , apply one exact multiplier update $\mu_1 = \mu_0 + c x_1$ after minimizing $L_c(\cdot, \mu_0)$.

2 $Q'_c(x) = 1 + cx = 0$, so

$x_c = -1/c \neq 0$ for any finite c .

3 $\partial_x L_c(x, \mu) = 1 + \mu + cx$. Hence

$$x_c(\mu) = -\frac{1+\mu}{c},$$

$$x_c(\bar{\mu}) = \bar{x} \quad \forall c > 0.$$

4 With $x_1 = -(1 + \mu_0)/c$,

$$\mu_1 = \mu_0 + cX_1 = -1 = \bar{\mu}.$$

At the following primal minimization, $x_2 = 0$.

Exercise 2: projection onto a line

Exercise

Consider

$$\min_{(x,y) \in \mathbb{R}^2} \frac{1}{2}(x^2 + (y-1)^2), \quad h(x,y) = y - x = 0.$$

For $c > 0$ and $\mu \in \mathbb{R}$,

$$L_c(x, y, \mu) = \frac{1}{2}(x^2 + (y-1)^2) + \mu(y-x) + \frac{c}{2}(y-x)^2.$$

- 1 Find the constrained solution and its KKT multiplier.
- 2 Determine the unique minimizer $(x_c(\mu), y_c(\mu))$.
- 3 Prove that the exact multiplier gives the constrained solution for every $c > 0$.
- 4 Compute $\mu^+ = \mu + c(y_c(\mu) - x_c(\mu))$. How does the multiplier error change?

$$(\bar{x}, \bar{y}) = (1/2, 1/2).$$
$$\begin{cases} (1+c)x - cy = \mu, \\ -cx + (1+c)y = 1 - \mu. \end{cases}$$
$$\begin{pmatrix} x_c(\mu) \\ y_c(\mu) \end{pmatrix} = \frac{1}{1+2c} \begin{pmatrix} c+\mu \\ 1+c-\mu \end{pmatrix}.$$

Exercise 6: solution (2/2)

3. For $\mu = \bar{\mu} = 1/2$,

$$(x_c(\bar{\mu}), y_c(\bar{\mu})) = (1/2, 1/2) \quad \forall c > 0.$$

For an approximate multiplier,

$$\|(x_c(\mu), y_c(\mu)) - (\bar{x}, \bar{y})\| = \frac{\sqrt{2} |\mu - 1/2|}{1 + 2c}.$$

4. Since

$$y_c(\mu) - x_c(\mu) = \frac{1 - 2\mu}{1 + 2c},$$

the multiplier update yields

$$\mu^+ = \mu + c \frac{1 - 2\mu}{1 + 2c} = \frac{\mu + c}{1 + 2c}.$$

Therefore

$$\mu^+ - \bar{\mu} = \frac{\mu - \bar{\mu}}{1 + 2c}.$$

With fixed $c > 0$, the multiplier error contracts geometrically.

Exercise 7: is any positive c sufficient?

Exercise

Consider the nonconvex problem

$$\min_{x \in \mathbb{R}} (x - x^2), \quad x = 0.$$

- 1 Determine the solution $\bar{x}$ and the KKT multiplier $\bar{\mu}$.
- 2 Write $L_c(x, \bar{\mu})$ and determine the values of c for which $\bar{x}$ is a strict local minimizer.
- 3 What happens when $c = 2$ and when $0 < c < 2$?
- 4 Explain why this does not contradict the general convex result.

- 2 The augmented Lagrangian at the exact multiplier is

Thus

3 If $c = 2$, $L_c(\cdot, -1) \equiv 0$: the minimizer is not strict. If $0 < c < 2$, $L_c(\cdot, -1)$ is unbounded below.

- The multiplier removes the need for $c \rightarrow \infty$, but a finite positive threshold may still be necessary.