# Continuous optimization
# ENT 305

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# And to sum up the courses ...

| | Necessary conditions | Sufficient conditions |
|---|---|---|
| Abstract formulation (exist.) | | if $K$ compact, $f \in C^0(K)$ then at least one solution |
| | | if $K$ closed, $f \in C^0(K)$, coercive then at least one solution |

| | Necessary conditions | Sufficient conditions |
|---|---|---|
| No constraints $K = \mathbb{R}^d$ (opt.) | if $\overline{x}$ local sol., $f \in C^2(K)$ then, $D^2 f(\overline{x})$ is positive semi-def. | if $f \in C^2(K)$, $\nabla f(\overline{x}) = 0$, $D^2 f(\overline{x})$ positive def. then $\overline{x}$ local sol. |
| Affine constraints | $\overline{x}$ local sol. then KKT | $f$ convex, then KKT=global sol. |
| Non-linear constraints | $\overline{x}$ local sol., LICQ then KKT | $f$ convex, $h$ affine, $g$ convex, then KKT=global sol. |

# And to sum up the courses ...

| | Necessary conditions | Sufficient conditions |
|---|---|---|
| Abstract formulation (exist.) | | if $K$ compact, $f \in C^0(K)$ then at least one solution |
| | | if $K$ closed, $f \in C^0(K)$, coercive then at least one solution |

| | Find a local solution |
|---|---|
| No constraints | Gradient Descent |
| Affine constraints | Penalty methods |
| Non-linear constraints | |

# Introduction

*Aim of the lecture:* a general presentation of one numerical methods for constrained optimization.

- **Penalty methods** $\rightsquigarrow$ equality constraints
- **Projected gradient methods** $\rightsquigarrow$ inequality constraints

well suited if constraints projection is possible and easy to compute.

*Reference:*

Nocedal and Wright. Numerical optimization. Springer Science and Business Media, 2006.

Boyd and Vandenberghe. Convex Optimization. Cambridge University Press, 2004.

# Quadratic penalization

We consider in this section

$$\inf_{x\in\mathbb{R}^n} f(x), \quad \text{subject to: } h(x) = 0, \qquad (P)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are given and "smooth".

*A general difficulty:* we need to cope with **two general goals**:

- Minimizing $f$
- Ensuring the feasibility of $x$.

When designing a numerical method, the question arises:
Given an iterate $x_k$, should we look for $x_{k+1}$ so that

$$f(x_{k+1}) < f(x_k) \quad \text{or} \quad \|h(x_{k+1})\| < \|h(x_k)\| \quad ?$$

# Quadratic penalization

*Main idea:* combining the two objectives into a single one.
Given a real number $c \geq 0$, consider the **penalty problem:**

$$\inf_{x \in \mathbb{R}^n} Q_c(x) := f(x) + \frac{c}{2}\|h(x)\|^2. \qquad (P_c)$$

*A rough statement:* if $c$ is large, $(P)$ and $(P_c)$ are "almost" equivalent.

*Big advantage of the approach:* numerical **methods of unconstrained optimization** can be employed for solving $(P_c)$.

# Quadratic penalization

**Exercise.**

Consider the problem:

$$\inf_{x \in \mathbb{R}} x, \quad \text{subject to: } x = 0.$$

1. What is the solution $\bar{x}$ to the problem?
2. Calculate the solution $x_c$ to the corresponding penalized problem $P_c$.
3. Verify that $x_c \underset{c \to +\infty}{\longrightarrow} \bar{x}$.

# Quadratic penalization

*Solution.*

1. Obviously $\bar{x} = 0$, since 0 is the unique feasible point of the problem.

2. Let $c > 0$. We have $Q_c(x) = x + \frac{c}{2}x^2$ and $\nabla Q_c(x) = 1 + cx$. Therefore,
$$\nabla Q_c(x) = 0 \iff x = -\frac{1}{c}.$$
Since $Q_c$ is convex, $x_c := -1/c$ is the unique solution of $(P_c)$.

3. Obviously
$$x_c = -1/c \underset{c \to \infty}{\longrightarrow} 0 = \bar{x}.$$

# Quadratic penalization

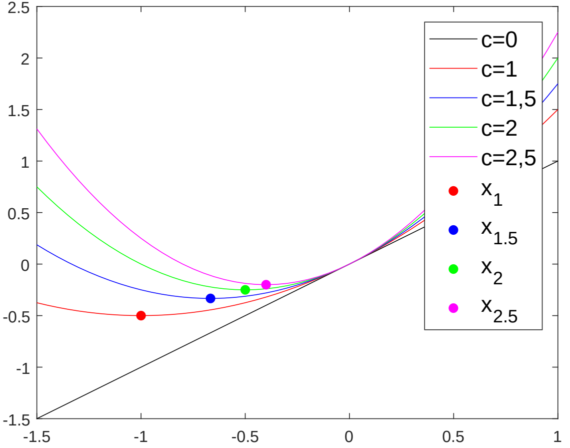


Figure: Graph of $Q_c$, for various values of $c$

# Quadratic penalization

**Lemma 1**

*Let $c_k \to \infty$. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$. Assume that*

- *For all $k \in \mathbb{N}$, $x_k$ is the* **solution** *to $(P_{c_k})$.*
- *The sequence $(x_k)_{k\in\mathbb{N}}$* **converges**, *let $\bar{x}$ denote the limit.*
- *There exists $\tilde{x}$ such that $h(\tilde{x}) = 0$.*

*Then, $\bar{x}$ is a* **solution** *to the original constrained problem $(P)$.*

*Proof.* Step 1. Let $x$ be a feasible point (that is, $h(x) = 0$). Then,

$$Q_{c_k}(x) = f(x) + \frac{c_k}{2}\|h(x)\|^2 = f(x).$$

In particular, $Q_{c_k}(\tilde{x}) = f(\tilde{x})$.

# Quadratic penalization

Step 2: $\bar{x}$ is feasible. For all $k \in \mathbb{N}$, we have

$$\begin{aligned} c_k \|h(x_k)\|^2 &= Q_{c_k}(x_k) - f(x_k) \\ &\leq Q_{c_k}(\tilde{x}) - f(x_k) && [\textit{Optimality of } x_k] \\ &= f(\tilde{x}) - f(x_k). && [\textit{Equality of Step 1}] \end{aligned}$$

Since $f(x_k) \to f(\bar{x})$, the sequence $(f(x_k))_{k \in \mathbb{N}}$ is bounded. Therefore, there exist $M > 0$ such that $c_k \|h(x_k)\|^2 \leq M$. Thus

$$\|h(x_k)\| \leq \sqrt{M/c_k}, \quad \forall k \in \mathbb{N}.$$

Passing to the limit, we get $\|h(\bar{x})\| \leq 0$. Thus $\bar{x}$ is **feasible**.

# Quadratic penalization

*Step 3.* Optimality of $\bar{x}$. Let $x$ be feasible. We have

$$\begin{aligned} f(x_k) &\leq f(x_k) + c_k \|h(x_k)\|^2 \\ &= Q_{c_k}(x_k) \\ &\leq Q_{c_k}(x) && [\textit{Optimality of } x_k] \\ &= f(x). && [\textit{Equality of Step 1}] \end{aligned}$$

Passing to the limit, we get

$$f(\bar{x}) \leq f(x).$$

Thus $\bar{x}$ is optimal.

# Quadratic penalization

The result of the lemma must be seen as an "ideal" situation.

**Difficulties** in practice:

- The problem $(P_c)$ **may not have a solution**, even if $(P)$ has a solution. Example:

$$\inf_{x \in \mathbb{R}} x^3, \quad \text{subject to: } x = 0.$$

- The sequence $(x_k)_{k \in \mathbb{N}}$ may not converge.
- The problem $(P_c)$ is **hard to solve** when $c$ is large, it is likely to be ill-conditioned (see next example).

# Quadratic penalization

*Example.* Consider:

$$\inf_{(x,y)\in\mathbb{R}^2} \frac{1}{2}\left(x^2+(y-1)^2\right), \quad \text{subject to: } x=y.$$

Projection problem of the point $(0,1)$ on the line $\{(x,y)\,|\,y=x\}$.

**Exercise.** Verify the following statements.

- Solution: $x^* = (0.5, 0.5)$.
- Solution of $P_c$, the penalty function, is: $\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \frac{1}{1+2c}\begin{pmatrix} c \\ 1+c \end{pmatrix}$.
- There exists a constant $M$ such that for all $c \geq 0$,

$$\|(x_c,y_c)-(\bar{x},\bar{y})\| \leq M/c.$$

# Quadratic penalization

*Solution.*

1. $\nabla f(x,y) = \begin{pmatrix} x \\ y-1 \end{pmatrix}$. The function $f$ is convex and thus, the global solution of the unconstrained version is $(0,1)$. With the constraints, we aim at minimizing $\frac{1}{2}(2x^2 - 2x + 1)$, and the unique solution is obviously $x = 0.5$.

2. $Q_c(x) = \frac{1}{2}(x^2 + (y-1)^2) + \frac{c}{2}(y-x)^2$ and $\nabla Q_c(x,y) = \begin{pmatrix} x - c(y-x) \\ y - 1 + c(y-x) \end{pmatrix}$, and since $Q_c$ is convex, the unique solution of $P_c$ is: $\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \dfrac{1}{1+2c} \begin{pmatrix} c \\ 1+c \end{pmatrix}$.

3. $\lim_{c\to\infty} \begin{pmatrix} x_c \\ y_c \end{pmatrix} = \lim_{c\to\infty} \frac{c}{c(1/c+2)} \begin{pmatrix} 1 \\ 1/c+1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
   $\|(x_c, y_c) - (0.5, 0.5)\|^2 = \frac{0.5}{(1+2c)^2} \Rightarrow \|(x_c, y_c) - (0.5, 0.5)\| = \frac{\sqrt{0.5}}{1+2c} \leq \frac{M}{c}$.
   Yet, $\nabla^2 Q(x,y) = \begin{pmatrix} 1+c & -c \\ -c & 1+c \end{pmatrix}$ which is ill-conditioned for large $c$. It yields difficulties with e.g. Newton algorithm $(\nabla^2 Q \cdot p = -\nabla Q)$ with abrupt function changes.

# Quadratic penalization

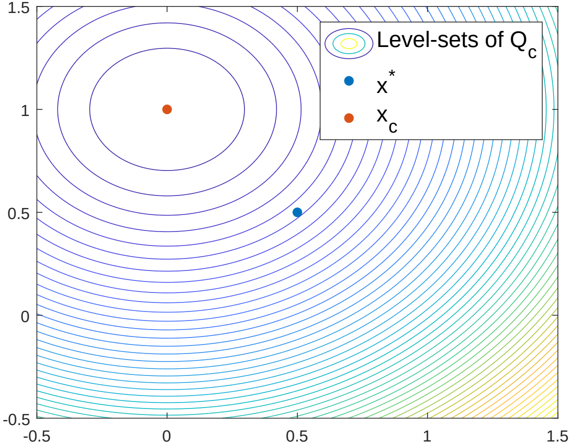


Figure: Graph of $Q_c$, for $c = 0$.

# Quadratic penalization

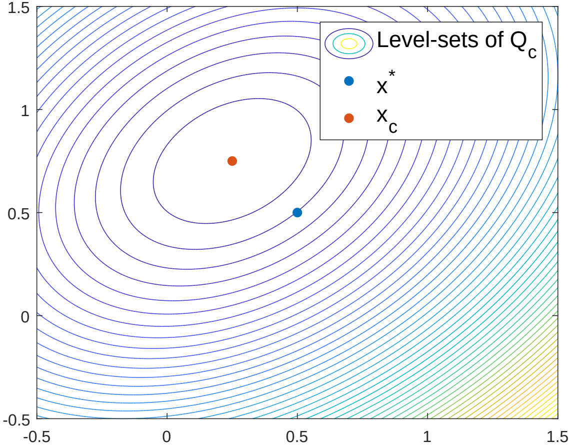


Figure: Graph of $Q_c$, for $c = 0.5$.

# Quadratic penalization

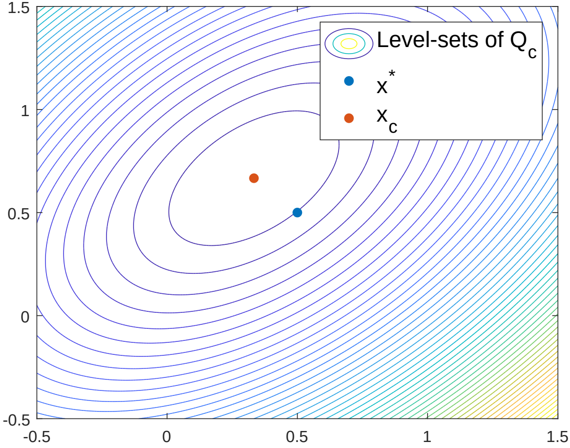


Figure: Graph of $Q_c$, for $c = 1$.

# Quadratic penalization

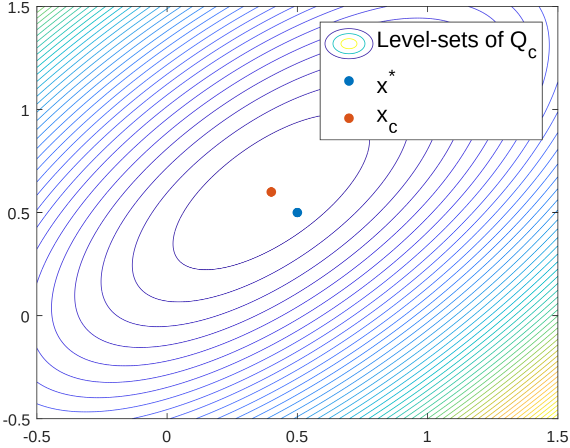


Figure: Graph of $Q_c$, for $c = 2$.

# Quadratic penalization

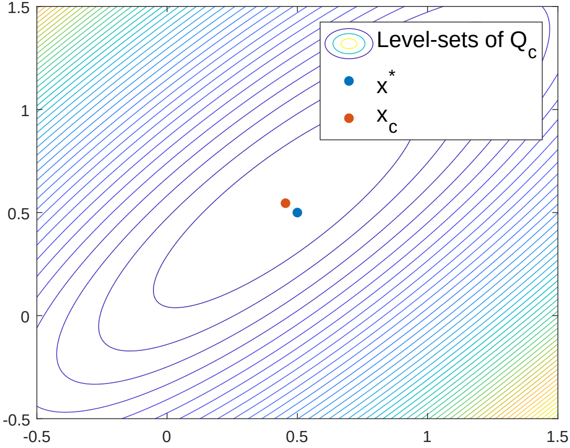


Figure: Graph of $Q_c$, for $c = 5$.

# Penalty algorithm

**General idea: increase the value of $c$ progressively, to mitigate the difficulty of minimizing $Q_c$.**

*Algorithm*:

1. Input: Choose $c_0 > 0$, starting point $x_0 \in \mathbb{R}^n$.
2. For $k = 1, ..., K - 1$, do
   - Solve $(P_{c_k})$ (e.g. with a gradient descent algorithm starting from $x_{k-1}$) and set $x_k$ the solution.
   - If $x_k$ is such that $h(x_k) = 0$, stop.
   - Otherwise choose $c_{k+1} > c_k$.

   End for.
3. Output: $x_K$.

# Penalty algorithm

$$Q_c(x) = f(x) + \frac{c}{2}\|h(x)\|^2$$

$$\begin{aligned}\nabla Q_c(x) &= \nabla f(x) + c\langle h(x), \nabla h(x)\rangle \\ &= \nabla L(x, ch(x))\end{aligned}$$

$$c_k h(x_k) \simeq \overline{\mu}$$

# Augmented Lagrangian

Unlike the penalty method, with the **augmented Lagrangian method** is not necessary to take $c \to \infty$ in order to solve the original constrained problem, avoiding ill-conditioning.

# Augmented Lagrangian

The two ideas of the **augmented Lagrangian method**:

1. Solving a penalty problem (like $(P_c)$) also yields an approximation of the Lagrange multiplier.
2. We can "improve" the penalty function $Q_c$ with the knowledge of that approximation.

*Algorithm*: at each iteration,

- the penalty parameter is increased
- the approximations $x_k$ of the solution and $\lambda_k$ of the Lagrange multiplier are improved.

# Augmented Lagrangian

Let $c > 0$. The **augmented Lagrangian** $L_c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is defined by

$$L_c(x, \mu) = f(x) + \langle \mu, h(x) \rangle + \frac{c}{2}\|h(x)\|^2.$$

$$\begin{aligned}\nabla L_c(x, \mu) &= \nabla f(x) + \langle \mu, \nabla h(x) \rangle + \langle ch(x), \nabla h(x) \rangle \\ &= \nabla L(x, \mu + ch(x))\end{aligned}$$

$$\mu_k + c_k h(x_k) \simeq \overline{\mu}$$

$$h(x_k) \simeq \frac{\overline{\mu} - \mu_k}{c_k}$$

$$\mu_{k+1} = \mu_k + c_k h(x_{k+1})$$

# Augmented Lagrangian

$$L_c(x,\mu) = f(x) + \langle \mu, h(x) \rangle + \frac{c}{2}\|h(x)\|^2.$$

We have

$$\begin{aligned} L_c(x,\mu) &= L(x,\mu) + \frac{c}{2}\|h(x)\|^2 \\ &= Q_c(x) + \langle \mu, h(x) \rangle \\ &= f(x) + \frac{c}{2}\|h(x) + \frac{\mu}{c}\|^2 - \frac{\|\mu\|^2}{2c} \end{aligned}$$

For a fixed $\lambda$, $L_c(\cdot,\mu)$ still serves as a **penalty function**. If $x_{c,\mu}$ minimizes $L_c(x,\mu)$ and if $c$ is very large, then

- $f(x_{c,\mu})$ is small
- $\frac{c}{2}\|h(x) + \frac{\mu}{c}\|^2$ is small $\to \|h(x) + \frac{\mu}{c}\|$ is very small
  $\to \|h(x)\|$ is very small.

# Augmented Lagrangian

The new **penalty problem:**

$$\inf_{x \in \mathbb{R}^n} L_c(x, \mu). \qquad (P_{c,\mu})$$

**Lemma 2**

*Let $\bar{x}$ be a local minimizer of $(P)$. Under technical assumptions, there exists $\bar{\mu}$ and $\bar{c} \geq 0$ such that for all $c > \bar{c}$,*

- *the* **KKT conditions** *hold true*
- *$\bar{x}$ is a* **local solution** *to $(P_{c,\bar{\mu}})$.*

# Reminders

| | Necessary conditions | Sufficient conditions |
|---|---|---|
| Abstract formulation (exist.) | | if $K$ compact, $f \in C^0(K)$ then at least one solution |
| | | if $K$ closed, $f \in C^0(K)$, coercive then at least one solution |

| | Necessary conditions | Sufficient conditions |
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| No constraints $K = \mathbb{R}^d$ (opt.) | if $\overline{x}$ local sol., $f \in C^2(K)$ then, $D^2 f(\overline{x})$ is positive semi-def. | if $f \in C^2(K)$, $\nabla f(\overline{x}) = 0$, $D^2 f(\overline{x})$ positive def. then $\overline{x}$ local sol. |
| Affine constraints | $\overline{x}$ local sol. then KKT | $f$ convex, then KKT=global sol. |
| Non-linear constraints | $\overline{x}$ local sol., LICQ then KKT | $f$ convex, $h$ affine, $g$ convex, then KKT=global sol. |

# Augmented Lagrangian

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## Lemma 3

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- *the* **KKT conditions** *hold true*
- *$\bar{x}$ is a* **local solution** *to $(P_{c,\bar{\mu}})$.*

*Idea of proof.* We have

$$\nabla L_c(\bar{x}, \bar{\mu}) = \nabla L(\bar{x}, \bar{\mu} + ch(\bar{x})) = \nabla L(\bar{x}, \bar{\mu}) = 0.$$

$$\nabla^2 L_c(\bar{x}, \bar{\mu}) = \nabla^2 L(\bar{x}, \bar{\mu}) + c\langle \nabla h(\bar{x}), \nabla h(\bar{x})\rangle$$

For $c$ large enough, $\nabla^2 L_c(\bar{x}, \bar{\mu})$ is positive definite.
Therefore, $\bar{x}$ is a local solution.

# Augmented Lagrangian

*Example 1.* Consider $\inf_{x\in\mathbb{R}} x - x^2$, subject to: $x = 0$.

**Exercise.**

- Write the Lagrangian formulation and find the Lagrangian multiplier.
- Does KKT holds for $\bar{x} = 0$?
- Write the augmented Lagrangian $(P_{c,\bar{\mu}})$ and show that $\bar{x}$ is a local solution to $(P_{c,\bar{\mu}})$ if $c > \bar{c}$.

# Augmented Lagrangian

*Example 1.* Consider $\inf_{x \in \mathbb{R}} x - x^2$, subject to: $x = 0$.

- Solution $\bar{x} = 0$.
- Lagrangian $L(x, \mu) = x - x^2 + \mu x$. We have

$$\nabla L(\bar{x}, \mu) = 1 - 2\bar{x} + \mu = 1 + \mu \implies \bar{\mu} = -1.$$

- Augmented lagrangian:

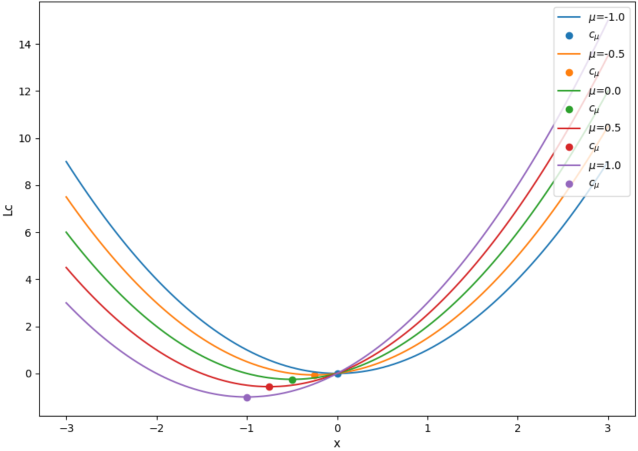
$$L_c(x, \mu) = x - x^2 + \mu x + \frac{c}{2} x^2 = (1 + \mu)x + \left(\frac{c}{2} - 1\right)x^2.$$

If $c > \bar{c} := 2$, $L_c(\cdot, \mu)$ has a unique minimizer

$$x_{c,\mu} = \frac{\mu + 1}{2 - c} = \frac{\mu - \bar{\mu}}{2 - c}.$$

In particular, $x_{c,\bar{\mu}} = \bar{x}$.

# Augmented Lagrangian



# Quadratic penalization

*Example 2.* Consider:

$$\inf_{(x,y)\in\mathbb{R}^2} \frac{1}{2}\big(x^2 + (y-1)^2\big), \quad \text{subject to: } x = y.$$

**Projection problem** of the point $(0,1)$ on the line $\{(x,y) \,|\, y = x\}$.

**Exercise.** Verify the following statements.

- Solution: $(\bar{x},\bar{y}) = (0.5, 0.5)$, $\bar{\mu} = 0.5$.
- Solution of $(P_{c,\mu})$ (aug. lagrangian): $\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \frac{1}{1+2c} \begin{pmatrix} c+\mu \\ 1+c-\mu \end{pmatrix}$.
- There exists a constant $M$ such that for all $c > 0$,

$$\|(x_c, y_c) - (\bar{x},\bar{y})\| \leq M|\bar{\mu} - \mu|/c.$$

# Quadratic penalization

*Solution.*

1.
- The function $f$ is convex: the global solution of the unconstrained version is given by stationarity:
  $\nabla f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which can be rewritten $\begin{pmatrix} x \\ y-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
  We find: $x = 0$ and $y - 1 = 0$ and thus, the global solution of the unconstrained version is $(0, 1)$.
- With the constraints $(y = x)$, we can replace $y$ by $x$ in the objective function $f$: we aim at minimizing $f(x) = \frac{1}{2}\left(2x^2 - 2x + 1\right)$.
  Again, $f$ is convex so the global solution is given by the point satisfying stationarity: $\nabla f(x) = 2x - 1 = 0$. We find the unique solution $\bar{x} = 0.5$.
- To find the Lagrange multiplyer, we replace in the Lagrangian gradient, $\bar{x}$ and $\bar{y}$ by 0.5:
  $L(\bar{x},\bar{y},\bar{\mu}) = f(\bar{x}) + \bar{\mu}(\bar{y} - \bar{x})$, so
  $\nabla L(\bar{x},\bar{y},\bar{\mu}) = \nabla f(\bar{x}) + \bar{\mu}\nabla h(\bar{x},\bar{y}) = \begin{pmatrix} \bar{x} - \bar{\mu} \\ \bar{y} - 1 + \bar{\mu} \end{pmatrix}$ and by stationarity $\nabla L(\bar{x},\bar{y},\bar{\mu}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ implies that $\bar{x} - \bar{\mu} = 0$ and $\bar{y} - 1 + \bar{\mu} = 0$. We find $\bar{\mu} = 0.5$.

# Quadratic penalization

2.

$$L_{c,\mu}(x,y) = f(x,y) + \frac{c}{2}h(x,y)^2 + \mu h(x,y)$$
$$= \frac{1}{2}(x^2 + (y-1)^2) + \frac{c}{2}(y-x)^2 + \mu(y-x)$$

(can be rewritten $\frac{1}{2}(x^2+(y-1)^2) + \frac{c}{2}(x-y)^2 + \mu(x-y)$) and $\nabla L_{c,\mu}(x,y) = \begin{pmatrix} x - c(y-x) - \mu \\ y - 1 + c(y-x) + \mu \end{pmatrix}$, and since $L_{c,\mu}$ is convex, the unique solution of $(P_{c,\mu})$ is the solution of stationarity condition:

$$\nabla L_{c,\mu}(x,y) = \begin{pmatrix} x - c(y-x) - \mu \\ y - 1 + c(y-x) + \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It gives two equations: $x - c(y-x) - \mu = 0$ and $y - 1 + c(y-x) + \mu = 0$ Adding the two together, we find $x + y - 1 = 0$, and thus $x = 1 - y$ or $y = 1 - x$.

In the first equation, we replace $y$ by $1-x$:

$x - c(1-x-x) - \mu = x(1+2c) - c - \mu = 0$, and thus, $x = \frac{c+\mu}{1+2c}$ and in the second one, we replace $x$ by $1-y$:

$y - 1 + c(y-1+y) + \mu = y(1+2c) - c - 1 + \mu = 0$, and thus,

$y = \frac{1+c-\mu}{1+2c}$. So, $\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \frac{1}{1+2c}\begin{pmatrix} c+\mu \\ 1+c-\mu \end{pmatrix}$.

# Quadratic penalization

3. $\|(a,b)\|^2 = a^2 + b^2$ (for euclidean norm)

$$\begin{aligned}
\|(x_c, y_c) - (\bar{x}, \bar{y})\|^2 &= (\frac{c+\mu}{1+2c} - \bar{x})^2 + (\frac{1+c-\mu}{1+2c} - \bar{y})^2 \\
&= (\frac{c+\mu}{1+2c} - 0.5)^2 + (\frac{1+c-\mu}{1+2c} - 0.5)^2 \\
&= \frac{1}{(1+2c)^2}((c+\mu-0.5-c)^2 + (1+c-\mu-0.5-c)^2) \\
&= \frac{1}{(1+2c)^2}((\mu-0.5)^2 + (0.5-\mu)^2) \\
&= \frac{2(\mu-0.5)^2}{(1+2c)^2} \\
&= \frac{2(\mu-\bar{\mu})^2}{(1+2c)^2}
\end{aligned}$$

$$\|(x_c, y_c) - (0.5, 0.5)\| = \frac{\sqrt{2}}{1+2c}|\mu - \bar{\mu}| \leq \frac{M|\mu-\bar{\mu}|}{c}.$$

# Augmented Lagrangian

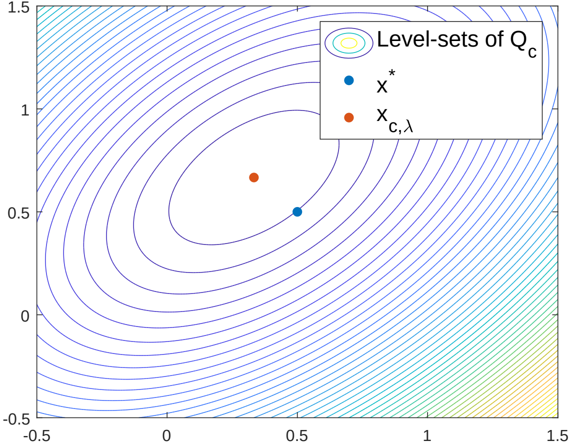



Figure: Level-sets $L_c(\cdot, \mu)$, for $c = 1$ and $\mu = 0$.

# Augmented Lagrangian

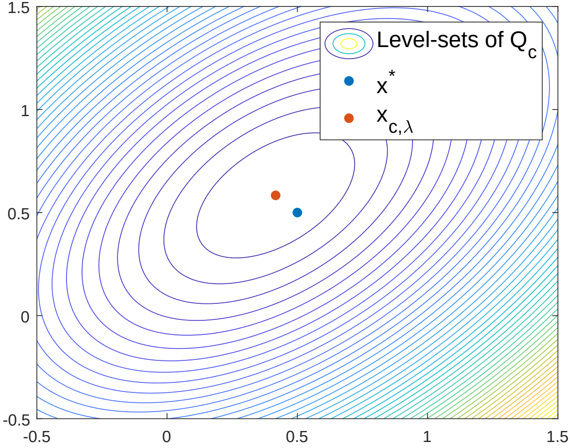



Figure: Level-sets $L_c(\cdot, \mu)$, for $c = 1$ and $\mu = 0,25$.

# Augmented Lagrangian

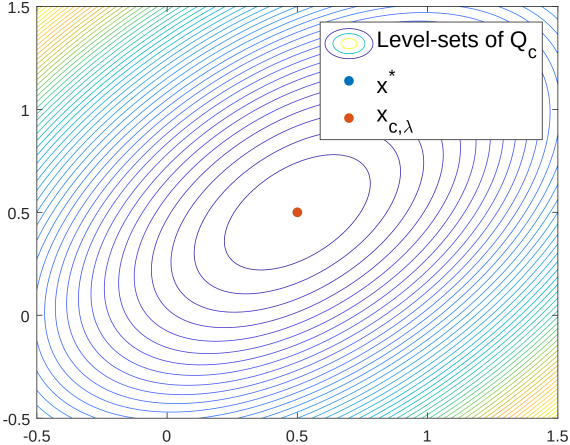



Figure: Level-sets $L_c(\cdot, \mu)$, for $c = 1$ and $\mu = 0, 5$.

# Augmented Lagrangian

*Algorithm.*

1. Input:
   - Initial point and multipliers $(x_0, \mu_0) \in \mathbb{R}^n \times \mathbb{R}^m$
   - Initial penalty parameter $c_0 > 0$, initial tolerance $\varepsilon_0 > 0$
   - Tolerance $\varepsilon > 0$.
2. Set $k = 0$.
3. While $\|D_x L(x_k, \mu_k)\| > \varepsilon$ and $\|h(x_k)\| > \varepsilon$,
   - Find $x_{k+1}$ such that $\|D_x L_{c_k}(x_{k+1}, \mu_k)\| \leq \varepsilon_k$.
   - If $\|h(x_{k+1})\|$ is small, set $\mu_{k+1} = \mu_k + c_k h(x_{k+1})$. Reduce $\varepsilon_k$.
   - Otherwise, increase $c_k$.
   - Set $k = k + 1$.

   End while.
4. Output $(x_k, \lambda_k)$.

# Lagrangian decomposition

*Main ideas* of **Lagrangian decomposition** methods:

- We take $c = 0$ in the augmented Lagrangian. At iterate $k$, given an approximation $\mu_k$ of the Lagrange multiplier, we solve

$$\inf_{x \in \mathbb{R}^n} L(x, \mu_k). \qquad (P_x)$$

where $\mu_k$ is found with the following maximization

$$\sup_{\mu \in \mathbb{R}^m} L(x, \mu)$$

Since $\nabla_\mu L(x, \mu) = h(x)$, this maximization is solved by iterating with an **ascent gradient step** to approximate the solution of $h(x) = 0$:

- Given a solution $x_{k+1}$, the Lagrange multiplier is updated by

$$\mu_{k+1} = \mu_k + \alpha h(x_{k+1}),$$

where $\alpha > 0 \to$ **Uzawa's algorithm**.

# Lagrangian decomposition

*Remarks*.

- Convergence of such methods can be established only under **convexity assumptions**.
- The stepsize $\alpha > 0$ must in general be small enough to ensure convergence. Instead of a fixed stepsize, one can use

$$\lambda_{k+1} = \lambda_k + \alpha_k g(x_{k+1}),$$

- One may consider instead of the primal problem $(P)$ the dual problem

$$d^* := \sup_{\mu \in \mathbb{R}^m} \inf_{x \in \mathbb{R}^n} L(x, \mu) \qquad (P_{\mu_k})$$

and we have $p^* \geq d^*$.

# Lagrangian decomposition

*Main advantage* of Lagrangian decomposition: very often the minimization of $L$ can be “parallelized”.

*Standard case:* additive constraints.

- Consider

$$\inf_{(x_1,x_2)\in X_1\times X_2} f_1(x_1)+f_2(x_2), \quad \text{subject to: } h_1(x_1)+h_2(x_2)=d,$$

  where $f_1$, $f_2$, $X_1$, $X_2$, $h_1$, $h_2$, and $d$ are given.
- Lagrangian:

$$\begin{aligned} L(x_1,x_2,\mu) &= f_1(x_1)+f_2(x_2)+\langle \mu, h_1(x_1)+h_2(x_2)-d\rangle \\ &= \Big[\underbrace{f_1(x_1)+\langle \mu, h_1(x_1)\rangle}_{=:L_1(x_1,\mu)}\Big]+\Big[\underbrace{f_2(x_2)+\langle \mu, h_2 x_2\rangle}_{=:L_2(x_2,\mu)}\Big]-\langle \mu, d\rangle. \end{aligned}$$

# Lagrangian decomposition

Given $\mu$, the minimization of $L(\cdot, \lambda)$ is **decomposed** into two subproblems:

$$\inf_{x_1 \in \mathbb{R}^{n_1}} L_1(x_1, \lambda) \qquad \text{and} \qquad \inf_{x_2 \in \mathbb{R}^{n_2}} L_2(x_2, \lambda),$$

which can be solved independently. Very often the two subproblems are **much easier** to solve than the original problem.

*Remark*. Straightforward **generalization** to the case

$$\inf_{\substack{x_1, \ldots, x_K \\ \in \mathbb{R}^{n_1} \times \ldots \mathbb{R}^{n_K}}} f_1(x_1) + \ldots + f_K(x_K), \quad \text{s.t.: } h_1(x_1) + \ldots + h_K(x_K) = K.$$

$\rightarrow$ Decomposition in $K$ subproblems (at each iteration).

# Lagrangian decomposition

1. **Application 1:** time decomposition.

- **Two production units**, with two independent production processes represented by the variables
  Problem:

$$\inf_{x \in \mathbb{R}^2} -\frac{x_1}{1+x_1} - \frac{x_2}{4+x_2}, \quad \text{s.t.} \left\{ x_1 + x_2 = d \right.$$

# Lagrangian decomposition

$$L(x, \mu) = -\frac{x_1}{1 + x_1} - \frac{x_2}{4 + x_2} + \mu(x_1 + x_2 - d).$$

If $x_1 + x_2 > d$, the engine must be rented for a longer time: the cost associated to constraints is increased. The incentive $\mu_k$ is too small, it must be increased.

If $x_1 + x_2 < d$, the cost associated to constraints is decreased. The incentive $\mu_k$ is too big, it must be decreased.
This is consistent with the formula

$$\mu_{k+1} = \mu_k + \alpha_k(x_1 + x_2 - d) \quad (4.1)$$

# Lagrangian decomposition

**Application 2:** stochastic decomposition.

- A production process is decomposed over two periods. A **random event** with two outcomes $\omega_1$ and $\omega_2$, with probabilities $p$ and $(1 - p)$, arises inbetween.
- Optimization variables:
  - $x_1$: decisions taken if outcome $\omega_1$ arises
  - $x_2$: decisions taken if outcome $\omega_2$ arises
  - $y$: decisions taken before the random event.

  *Example:* purchase of gas $y$ on a day-ahead market (that is, on a given day for the next one).
  Random event: temperature, which impacts consumption.

# Lagrangian decomposition

- Abstract problem:

$$\inf_{\substack{(x_1,x_2,y)\\(x_1,y)\in X\\(x_2,y)\in X}} pf(x_1,y,\omega_1)+(1-p)f(x_2,y,\omega_2).$$

- Equivalent problem (with non-anticipativity constraint):

$$\inf_{\substack{(x_1,x_2,y_1,y_2)\\(x_1,y_1)\in X\\(x_2,y_2)\in X}} pf(x_1,y_1,\omega_1)+(1-p)f(x_2,y_2,\omega_2), \quad \text{s.t. } y_2-y_1=0.$$

- Independent (w.r.t. randomness) sub-problems:

$$\inf_{(x_1,y_1)\in X_1} pf_1(x_1,y_1,\omega_1)+\mu_k y_1, \qquad \inf_{(x_2,y_2)\in X_2}(1-p)f_2(x_2,y_2,\omega_2)-\mu_k y_2.$$

1. Penalty methods for constrained optimization
   - Quadratic penalization
   - Augmented Lagrangian
   - Lagrangian decomposition

2. Projected gradient method
   - Projection
   - Method
   - Combination with penalty methods

# Projection

**Idea: Apply steepest descent method but project the path onto the constraints.** The projected gradient method uses a mapping called **projection** defined below.

## Lemma 4

*Let $K \subset \mathbb{R}^n$ be a non-empty, convex, and closed set. For all $x_0 \in \mathbb{R}^n$, there exists a* **unique solution** *to the problem*

$$\inf_{x \in \mathbb{R}^n} \|x - x_0\|^2, \quad \textit{subject to: } x \in K.$$

*It is called* **projection** *of $x_0$ on $K$, and denoted $Proj_K(x_0)$.*

*Remark.* The projection depends on the chosen norm $\| \cdot \|$. For simplicity, we consider the Euclidean norm.

# Projection

**Example 1:** projection on a cuboid.
Let $K$ be described by

$$K = \{x \in \mathbb{R}^n \,|\, \ell_i \leq x_i \leq u_i\},$$

where the coefficients $\ell_1,\dots,\ell_n \in \mathbb{R} \cup \{-\infty\}$ and $u_1,\dots,u_n \in \mathbb{R} \cup \{+\infty\}$ are given.

Let $x \in \mathbb{R}^n$, let $y = \mathrm{Proj}_K(x)$. Then

$$y_i = \min(\max(x_i, \ell_i), u_i), \quad \forall i = 1, \dots, n.$$

# Projection

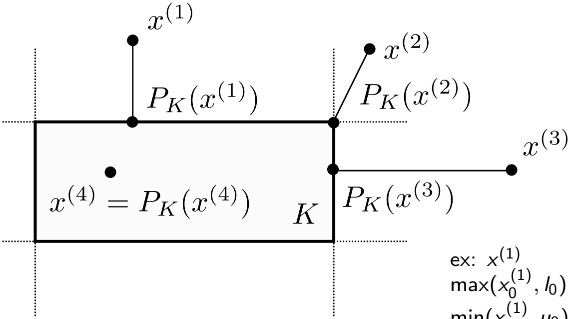



Figure: Projection on a cuboid.

ex: $x^{(1)}$
$\max(x_0^{(1)}, l_0) = x_0^{(1)}$
$\min(x_0^{(1)}, u_0) = x_0^{(1)}$

$\max(x_1^{(1)}, l_1) = x_1^{(1)}$
$\min(x_1^{(1)}, u_1) = u_1$

# Projection

**Example 2:** projection on a ball.
Let $K$ be described by

$$K = \left\{ x \in \mathbb{R}^n \mid \|x - x_C\| \leq R \right\},$$

where $x_C \in \mathbb{R}^n$ and $R \geq 0$ are given.

For all $x \in \mathbb{R}^n$,

$$\mathrm{Proj}_K(x) = x_C + \min\left(\|x - x_C\|, R\right) \frac{(x - x_C)}{\|x - x_C\|}.$$

# Projection

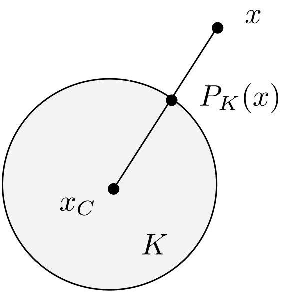



Figure: Projection on a ball.

# Projection

**Example 3:** cartesian product.
Let $K$ be given by

$$K = K_1 \times K_2,$$

where $K_1$ and $K_2$ are given non-empty closed and convex subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$.

Then for all $x = (x_1, x_2) \in \mathbb{R}^{n_1+n_2}$,

$$\mathrm{Proj}_K(x) = \Big(\mathrm{Proj}_{K_1}(x_1), \mathrm{Proj}_{K_2}(x_2)\Big).$$

# Method

*Optimization problem.* Consider

$$\inf_{x \in \mathbb{R}^n} f(x), \quad x \in K,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is given and differentiable and $K$ is a given non-empty **convex** and **closed** subset of $\mathbb{R}^n$.

*Numerical assumption:* $\mathrm{Proj}_K(\cdot)$ is **easy to compute**.

*Gradient descent algorithm.*

1. Input: $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$. Set $k = 0$.
2. While $\|\nabla f(x_k)\| \geq \varepsilon$, do
   - Find a descent direction $d_k$.
   - Find $\alpha_k > 0$ such that $f(x_k + \alpha_k d_k) < f(x_k)$.
   - Set $x_{k+1} = x_k + \alpha_k d_k$.
   - Set $k = k + 1$.
3. Output: $x_k$.

*Main idea:*
at iteration $k$, replace the search on the half line $\{x_k + \alpha_k d_k \,|\, \alpha \geq 0\}$ used in unconstrained optimization by a **search** on

$$\Big\{ \underbrace{\mathrm{Proj}_K(x_k + \alpha_k d_k)}_{=: x_{k+1}(\alpha_k)} \,|\, \alpha_k \geq 0 \Big\}.$$

# Combination with penalty methods

Consider the problem

$$\inf_{x\in\mathbb{R}^n} f(x), \quad \text{subject to:} \begin{cases} h_i(x) = 0 & \forall i \in \mathcal{E}, \\ g_i(x) \leq 0 & \forall i \in \mathcal{I}, \end{cases}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$, $h\colon \mathbb{R}^n \to \mathbb{R}^{m^1}$ and $g\colon \mathbb{R}^n \to \mathbb{R}^{m_2}$ are given.

Idea: Eliminate inequality constraints by slack variables. An **equivalent formulation** is

$$\inf_{\substack{x\in\mathbb{R}^n \\ y\in\mathbb{R}^m}} f(x), \quad \text{subject to:} \begin{cases} \Phi(x) - y = 0 \\ y \in K, \end{cases}$$

where: $\Phi_i(x) = \begin{cases} h_i(x), \ \forall i \in \mathcal{E}, \\ g_i(x), \ \forall i \in \mathcal{I}, \end{cases}$ and

$$K = \left\{ y \in \mathbb{R}^m \,\middle|\, \begin{cases} y_i = 0 & \forall i \in \mathcal{E} \\ y_i \leq 0 & \forall i \in \mathcal{I} \end{cases} \right\}.$$

# Combination with penalty methods

*Main idea:* projection on $K$ (a cuboid) is easy to compute.
Handle $y \in K$ with the projected gradient method.

*Algorithm.*

- At iteration $k$, the iterates $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^m$, $\mu_k \in \mathbb{R}^m$, and $c_k$ are given.
- Solve (approximately) the penalty problem:

$$\inf_{\substack{x \in \mathbb{R}^n \\ y \in \mathbb{R}^m}} L_{c_k}(x, y, \mu_k) := f(x) + \langle \mu_k, \Phi(x) - y \rangle + \frac{c_k}{2} \|\Phi(x) - y\|^2,$$

subject to: $y \in K$,

with the projected gradient method.
Use $(x_k, y_k)$ as a starting point.