

NAME:

ENT305A: Exam

24/10/2025

Elise Grosjean, ENSTA-Paris

Answers may be written in French or English.

Exercise 1 (Review of Course Material).

- State the Weierstrass theorem.
- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be of class $\mathcal{C}^2$ on $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$ be such that $\nabla f(x) = 0$. State a sufficient condition for x to be a local solution to the minimization problem:

$$\inf_{x \in \mathbb{R}^n} f(x).$$

Exercise 2 (KKT Conditions). Consider the following optimization problem:

$$\inf_{(x_1, x_2) \in \mathbb{R}^2} f(x_1, x_2) := \frac{1}{2}x_1^2 - 4x_1 + 2x_2^2 - 8x_2.$$

1. For this question, consider only the equality constraint

$$(x_1 + x_2^2) - 3 = 0.$$

Write the expression for the associated augmented Lagrangian L_c .

2. We now modify the constraints; the problem becomes

$$\inf_{(x_1, x_2) \in \mathbb{R}^2} f(x_1, x_2) \quad \text{s.t.} \quad \begin{cases} (x_1 + x_2^2) - 3 \leq -y, \\ x_2 \leq 1 + \frac{1}{2}x_1, \\ x_2 \geq 0, \end{cases}$$

where $y \in \mathbb{R}$ is a parameter.

We first set $y = 0$ (for questions 2 through 5). Prove that a solution exists.

3. Write down the KKT conditions. Verify that the point $(2, 1)$ is feasible and satisfies these conditions.
4. State the LICQ condition for active inequality constraints (there are no equality constraints here). Does the point $(2, 1)$ satisfy this condition?
5. Is the point $(2, 1)$ a global solution to the problem?
6. The parameter y can now take any real value. What change in the optimal cost results from a variation δy in the parameter?

Exercise 3 (Pseudocode). Write pseudocode for a gradient descent algorithm.

Bonus: Add an Armijo rule for the step-size selection.

Exercise 4. Consider the following set in $\mathbb{R}^2$:

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 \leq 1 \text{ and } x_2^2 \leq 1\}.$$

1. What is the geometric shape of C ?
2. Let g_1 be the constraint associated with x_1 and g_2 the constraint associated with x_2 . Compute $\nabla^2 g_1$ and $\nabla^2 g_2$. Show that C is convex and closed.

3. Let A be an invertible real 2×2 matrix, and let b be a vector in $\mathbb{R}^2$. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f : x \mapsto \frac{1}{2} \|Ax - b\|_2^2,$$

and the optimization problem

$$\min f(x) \quad \text{s.t. } x \in C. \quad (\text{P})$$

4. Show that (P) admits a unique solution.
5. We now seek to determine the minimum of f over C in the case where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

- Find the solution to the unconstrained problem (P). Is this point feasible?
 - Assume that the violated constraint is active at the optimal point. Write down the KKT conditions.
 - Solve the resulting system assuming $|x_2| < 1$. Are the KKT conditions satisfied in this case?
 - Answer the same question for $|x_2| = 1$.
 - Conclude: What is the solution to the constrained minimization problem (P)?
6. Bonus: Write pseudocode for the projected gradient algorithm associated with (P).

Exercise 5 (Modeling and Optimization). A company supplies gas over $T = 2$ time periods ($t = 1, 2$) to $M = 2$ districts ($j = 1, 2$) from a single extraction site ($N = 1$). There are $K = 2$ permitted connections: the site can supply districts 1 and 2 (no other connections are allowed).

The company has a lossless storage tank with capacity $S_{\max}$. The initial and final storage levels are assumed to be zero: $s(0) = s(2) = 0$.

The production cost at time t is $a x(t)^2 + b x(t)$.

The decision variables are:

- $x(t)$: production at the site during period t ,
- $z_1(t), z_2(t)$: flows sent from the site to districts 1 and 2 during period t ,
- $y(t)$: flow sent from the site to the storage tank during period t ,
- $u_1(t), u_2(t)$: flows sent from the storage tank to districts 1 and 2 during period t ,
- $s(t)$: storage level at the *end* of period t .

Demand $D(t, j)$ must be met in every period and in every district. The parameters are:

	$D(t, j)$	$t = 1$	$t = 2$
$S_{\max} = 1,$	$a = 2,$	$b = 1,$	
	$j = 1$	1	0
	$j = 2$	1	1

1. Write the optimization problem that *minimizes the total production cost*.

Hints. The flow-balance constraints at the extraction site and the storage tank must be modeled, and demand must, of course, be satisfied.

2. Which algorithm could be used to solve this problem? Explain why.