Reduced basis methods for the resolution of parameter-dependent PDEs
MS13

Elise Grosjean
Ensta-Paris

Ensta-Paris
Institut Polytechnique de Paris

Introduction

◇ **Main goal**
The objective of RBM is to find **very quickly accurate approximations** of parameter-dependent functions of the generic form

$$u : \Omega \times \mathcal{G} \to \mathbb{R},$$

– $\Omega \in \mathbb{R}^d$: the spatial domain,
– $\mathcal{G} \subset \mathbb{R}^{N_p}$: the parameter domain, with N_p the number of parameters.
$\mu = (\mu_1, \dots, \mu_{N_p}) \in \mathcal{G}$: the varying parameter.

$$\mathcal{L}(\mu)(u(\mu)) = F(\mu), \text{ in } \Omega,$$
$$+ \text{ boundary conditions on } \partial\Omega.$$

Reduced Basis Methods (RBM)

$PDE : \boldsymbol{\mu} \to u(\boldsymbol{\mu})$

$\boldsymbol{\mu} \in \mathcal{G}$: Parameter

$u(\boldsymbol{\mu})$: Solution

$$\begin{cases} -\nabla \cdot \big(a(\boldsymbol{\mu})\nabla u\big) = f(\boldsymbol{\mu}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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$\boldsymbol{\mu} \in \mathcal{G}$: Parameter

$$u_h(\boldsymbol{\mu};\ \mathbf{x}) = \sum_{i=1}^{\mathcal{N}} u_i(\boldsymbol{\mu}) w_i(\mathbf{x}),$$

$$\begin{cases} \text{where } \mathbf{u}(\boldsymbol{\mu}) = (u_1(\boldsymbol{\mu}), \ldots, u_{\mathcal{N}}(\boldsymbol{\mu}))^T \in \mathbb{R}^{\mathcal{N}} \\ \text{is the solution of a linear system} \\ \mathbf{A}(\boldsymbol{\mu})\mathbf{u}(\boldsymbol{\mu}) = \mathbf{f}(\boldsymbol{\mu}). \end{cases}$$

Convergence of the finite element method

Provided that *"sufficiently uniform"* meshes are used and $\mathbb{P}_1$ FE,

$$\|u_h - u\|_{H^1} = \mathcal{O}(h),$$

where h denotes the mesh size.

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Aim of the reduced basis methods (RBM)

Solve the PDE as quickly as possible when it has to be evaluated for many parameter values

Applications

- Real-time simulations
- Parametric studies
- ...

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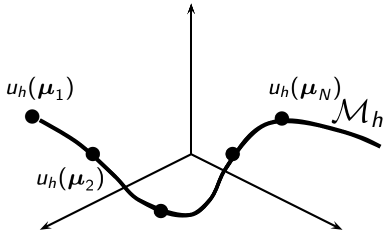

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Solution manifold: $\mathcal{M} = \{u(\boldsymbol{\mu})|\ \boldsymbol{\mu} \in \mathcal{G}\}$
HF solution manifold: $\mathcal{M}_h = \{u_h(\boldsymbol{\mu})|\ \boldsymbol{\mu} \in\ \mathcal{G}\}$

How can we reduce the manifold complexity?

Reduced space

V^N Reduced space

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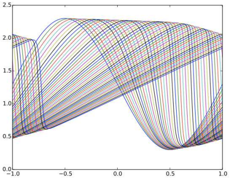

Fig. 1: Snapshots of the solution to the unsteady viscous Burger equation with $u_0 = \lambda$
$\lambda = 1.3$, $\nu = 4$, $\varepsilon = 0.04$

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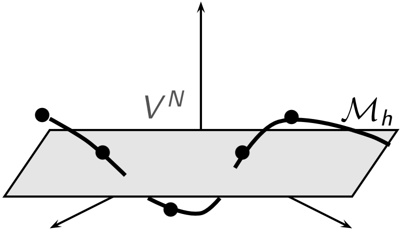
where $\mathbf{u}(\boldsymbol{\mu}) = (u_1(\boldsymbol{\mu}), \ldots, u_{\mathcal{N}}(\boldsymbol{\mu}))^T \in \mathbb{R}^{\mathcal{N}}$ is the solution of a linear system $\mathbf{A}(\boldsymbol{\mu})\mathbf{u}(\boldsymbol{\mu}) = \mathbf{f}(\boldsymbol{\mu})$.



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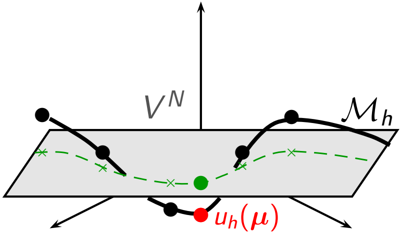

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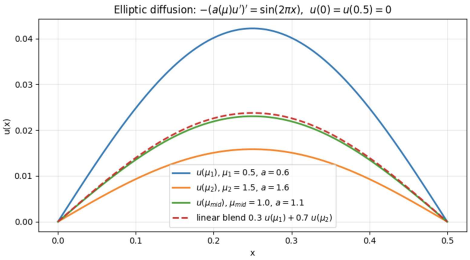

$$V_N = \mathrm{Span}\{u_1, u_2, \ldots u_N\}$$

where $u_1, \ldots, u_N =$ snapshots

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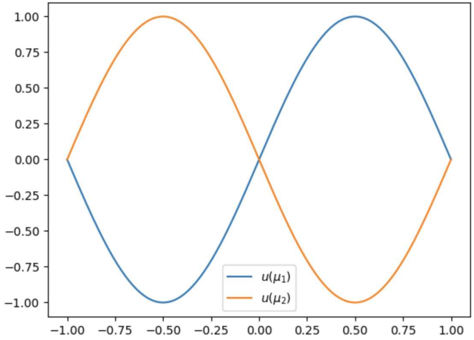

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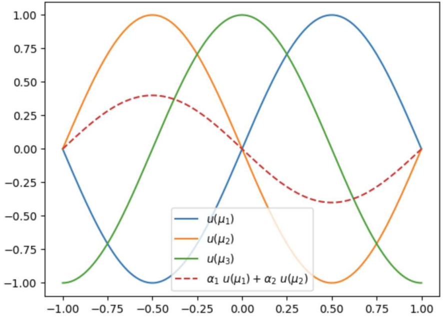

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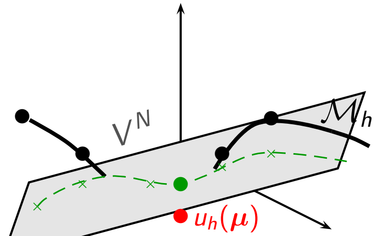

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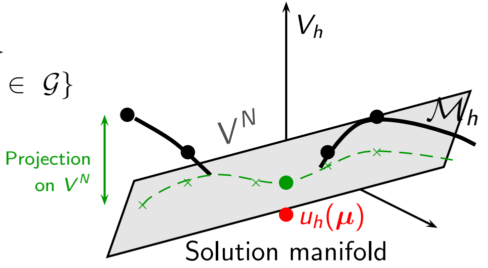
Projection on V^N:

$$\inf_{v_N \in V^N} \|u_h - v_N\|_{V_h}.$$

Kolmogorov N-width = error from the **linear** space that best fit the solution manifold:

$$d_N(\mathcal{M}_h, V_h) = \inf_{\substack{V^N \subset V_h \\ \dim(V^N)=N}} \ \sup_{u_h \in \mathcal{M}_h} \ \inf_{v_N \in V^N} \|u_h - v_N\|_{V_h}.$$



Exponential decay

$$\exists, \tau, C > 0,\ \ \forall N > 1, \quad d_N(\mathcal{M}, V) \leq Ce^{-}$$

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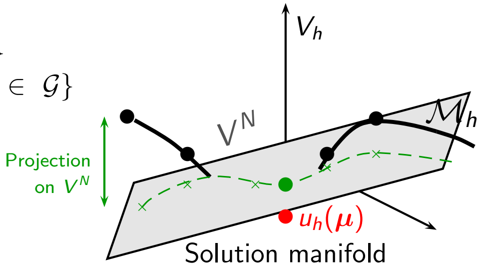

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$(\mathcal{M})$

Reduced Basis Methods

- ◇ **Offline** Construction of a reduced space V_N spanned by a reduced basis.
- ◇ **Online** Computation of the reduced coefficients $\boldsymbol{\alpha}$.

The optimal reduced space V^N may not be found
Two main algorithms to find approximated reduced spaces: **the Proper Orthogonal Decomposition (POD)** or **greedy algorithms**.

Reduced basis Galerkin approximation

Assume the weak formulation of the HF problem yields the discretized system

$$\mathbf{A}(\boldsymbol{\mu})\mathbf{u}(\boldsymbol{\mu}) = \mathbf{l}(\boldsymbol{\mu})$$

then

$$a\left(u_N(\boldsymbol{\mu}), v_N; \boldsymbol{\mu}\right) = \ell\left(v_N; \boldsymbol{\mu}\right)$$

gives a new system to solved:

$$\boxed{\mathbf{P}^T\mathbf{A}(\boldsymbol{\mu})\mathbf{P}\ \alpha(\boldsymbol{\mu}) = \mathbf{P}^T\mathbf{l}(\boldsymbol{\mu}),} \tag{G-RB}$$

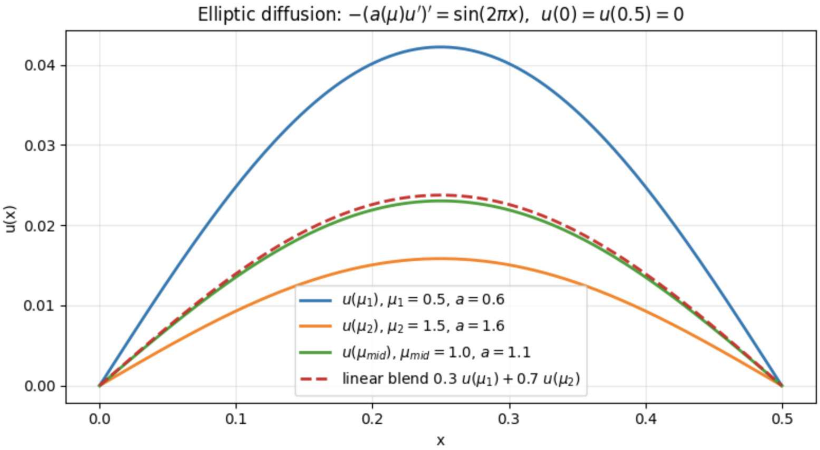
where $\mathbf{P} \in \mathbb{R}^{\mathcal{N}\times N}$. Now, we get a system where the inversion cost is in $\mathcal{O}(N^3)$ since dimensions : $\mathbf{P}^T\mathbf{A}(\boldsymbol{\mu})\mathbf{P} \in \mathbb{R}^{N\times N}$ and $\mathbf{P}^T b \in \mathbb{R}^N$!

Reduced basis Galerkin approximation

Assembling cost with the affine operators:
$\mathcal{O}(N^2 Q^a + N Q^l)$ with

$$\mathbf{P}^T\mathbf{A}(\boldsymbol{\mu})\mathbf{P} = \sum_{q=1}^{Q^a} \theta_q^a(\boldsymbol{\mu}) \underbrace{\mathbf{P}^T\mathbf{A}_q\mathbf{P}}_{\text{precomputed offline}} \quad , \quad \mathbf{P}^T\mathbf{l}(\boldsymbol{\mu}) = \sum_{q=1}^{Q^l} \theta_q^l(\boldsymbol{\mu}) \underbrace{\mathbf{P}^T\mathbf{l}_q}_{\text{precomputed offline}} \quad .$$

POD: Continuous version



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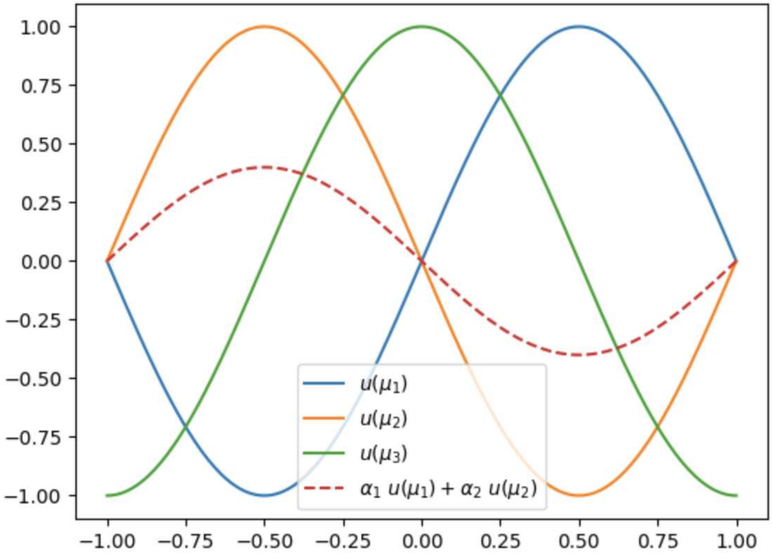


POD: Continuous version

How many snapshots do we need to represent our data?

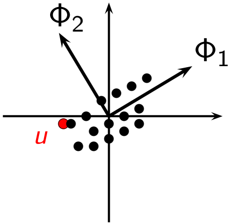
POD: Continuous version

Suppose we need M snapshots, the POD compresses our data by using $N \leq M$ basis functions!

POD: Continuous version

We want to approximate $u(\mathbf{x}, \boldsymbol{\mu})$ by $\sum_{k=1}^{N} a_k(\boldsymbol{\mu})\Phi_k(\mathbf{x})$.

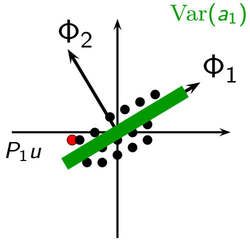
Let us consider $\boldsymbol{\mu}$ a random variable and u centered ($\mathbb{E}_{\boldsymbol{\mu}}[u] = 0$).
POD = PCA: We want to find the axes that best represent the data!



$$\min_{\|\Phi_i\|=1} \mathbb{E}[\|u - \sum_{k=1}^{N} a_k(\boldsymbol{\mu})\ \Phi_k\|^2].$$

POD: Continuous version

How do we find the reduced basis?



$$Var(a_1) = \mathbb{E}[a_1^2] - (\mathbb{E}[a_1])^2 = \mathbb{E}[a_1^2]$$

$$a_1 = (u, \Phi_1), \;\; \|\Phi_1\| = 1.$$

$$\boxed{\min_{\|\Phi_1\|=1} \mathbb{E}[\|u - (u,\Phi_1)\Phi_1\|^2] \text{ or } \max_{\|\Phi_1\|=1} \mathbb{E}[|(u,\Phi_1)|^2] \text{ or } C\Phi_1 = \lambda_1\Phi_1}$$

Snapshot POD

Spectral theorem (Compact Self-Adjoint Operator)
Let V be a separable Hilbert space and let $C : V \to V$ be a compact, positive, self-adjoint operator. Then there exists an orthonormal basis $\{\Phi_n\}_{n\in\mathbb{N}}$ of V and a sequence of real positive numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$,

$$C\Phi_n = \lambda_n \Phi_n.$$

Moreover, $\lim_{n\to\infty} \lambda_n = 0$.

$$\boxed{\mathbb{E}[(u,\Phi)u] = \lambda\Phi, \text{ i. e. } C\Phi = \lambda\Phi}.$$

One can prove that C is a positive linear compact self-adjoint operator: one unique solution equal to the largest eigenvalue of the problem!

$(C\Phi,\Phi) = \mathbb{E}[|(u,\Phi)|^2] = \lambda$

$\max_{\|\Phi_1\|=1} \mathbb{E}[|(u,\Phi_1)|^2] = \lambda_1$

POD energy error

$\lambda_1 >= \lambda_2 >= ... >= 0$

$$\boxed{\mathbb{E}[\|u - P_N u\|^2] = \sum_{k>N} \mathbb{E}[a_k^2] = \sum_{k>N} \lambda_k}$$

In fact, one can show that the more regularizing the operator C is, the faster its eigenvalues decay!

Link between regularity and eigenvalues

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Exercise

Let $A : L^2(0,1) \to L^2(0,1)$ be defined by $(Ax)(t) = \int_0^t x(s)\,ds$.

1. Determine the adjoint operator A^*.

2. Define $C = A^*A$. Show that C is a compact, positive, and self-adjoint operator on $L^2(0,1)$.

3. Let σ denote a singular value of A, and let $x \neq 0$ satisfy $Cx = A^*Ax = \sigma^2 x$.

(a) Show that A^*Ax is twice differentiable and compute $(A^*Ax)''(t)$.

(b) Deduce that x satisfies the differential equation $x(t) + \sigma^2 x''(t) = 0$.

4. Using the appropriate boundary conditions, determine the general form of the eigenfunctions $x(t)$.

5. Deduce the explicit expression of the singular values σ_n of A and determine their asymptotic behaviour as $n \to \infty$.

Link between regularity and eigenvalues

Solution

Let $A : L^2(0,1) \to L^2(0,1)$ be defined by $(Ax)(t) = \int_0^t x(s)\,ds$.

1. Adjoint A^*. For $x, y \in L^2(0,1)$,
$(Ax, y) = \int_0^1 \left(\int_0^t x(s)\,ds\right) y(t)\,dt = \int_0^1 x(s) \left(\int_s^1 y(t)\,dt\right)\,ds = (x, A^* y)$,

2.

Self-adjointness: $C = A^* A$ is self-adjoint since $(A^* A)^* = A^* (A^*)^* = A^* A$.

Positivity: $(Cx, x) = (A^* Ax, x) = (Ax, Ax) = \|Ax\|_{L^2}^2 \geq 0$.

Compactness: A maps $L^2(0,1)$ continuously into $H^1(0,1)$, since $Ax \in H^1(0,1)$ and

$$(Ax)' = x \quad \text{in } L^2(0,1), \qquad \|Ax\|_{H^1}^2 = \|Ax\|_{L^2}^2 + \|x\|_{L^2}^2 \leq C\|x\|_{L^2}^2.$$

The embedding $H^1(0,1) \hookrightarrow L^2(0,1)$ is compact (Rellich), hence A is compact and consequently $C = A^* A$ is compact.

Link between regularity and eigenvalues

Solution

3. Spectral problem. Let σ be a singular value of A and let $x \neq 0$ satisfy
$Cx = A^*Ax = \sigma^2 x$.
Set $y = A^*Ax$. Since $(A^*y)(t) = \int_t^1 y(s)\,ds$, $(A^*y)'(t) = -y(t)$ (p.p).
Hence $(A^*Ax)'(t) = -(Ax)(t)$ (p.p), $(A^*Ax)''(t) = -x(t)$
On the other hand, $A^*Ax = \sigma^2 x$ implies $(A^*Ax)'' = \sigma^2 x''$, thus
$\sigma^2 x''(t) = -x(t) \quad \Longleftrightarrow \quad x(t) + \sigma^2 x''(t) = 0.$

Boundary conditions. From $A^*Ax = \sigma^2 x$:
at $t = 1$, $(A^*Ax)(1) = \int_1^1 (Ax)(s)\,ds = 0 = \sigma^2 x(1) \quad \Longrightarrow \quad x(1) = 0$. Also,
$(A^*Ax)'(0) = -(Ax)(0) = 0 = \sigma^2 x'(0) \quad \Longrightarrow \quad x'(0) = 0$.

4. Eigenfunctions. Let $\omega = \sigma^{-1}$. The ODE becomes $x'' + \omega^2 x = 0$, so
$x(t) = A\cos(\omega t) + B\sin(\omega t)$. The condition $x'(0) = 0$ gives $B = 0$. Then $x(1) = 0$ yields
$A\cos(\omega) = 0 \quad \Longleftrightarrow \quad \omega = \frac{\pi}{2} + n\pi, \qquad n \in \mathbb{N}$. Thus the (normalized) eigenfunctions are

$$x_n(t) = A\cos\left(\left(n + \tfrac{1}{2}\right)\pi\, t\right).$$

Link between regularity and eigenvalues

Solution

5. Singular values and asymptotics. Since $\omega_n = \sigma_n^{-1} = \left(n + \frac{1}{2}\right)\pi$, we obtain $\sigma_n = \frac{1}{\left(n+\frac{1}{2}\right)\pi} = \frac{2}{(2n+1)\pi}$, $n \in \mathbb{N}$. Therefore, $\sigma_n \sim \frac{1}{\pi n}$ as $n \to \infty$. The eigenvalues of $C = A^*A$ are $\mu_n = \sigma_n^2$, hence $\mu_n = \frac{1}{\left(\left(n+\frac{1}{2}\right)\pi\right)^2} = \frac{4}{(2n+1)^2\pi^2} \sim \frac{1}{\pi^2 n^2}$.

Link between regularity and eigenvalues

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Exercise: Transport equation

Let $T_h : L^2(0,1) \to L^2(0,1)$ be the transport operator defined by $(T_h x)(t) = x(t-h)$, with periodic boundary conditions on $(0,1)$ and a fixed shift $h \in (0,1)$.

1. Show that T_h is a bounded linear operator on $L^2(0,1)$ and that $\|T_h x\|_{L^2} = \|x\|_{L^2}$.
2. Compute the adjoint T_h and show that $T_h^* = T_{-h}$.
3. Show that $T_h^* T_h = I$, where I is the identity operator.
4. Deduce that the singular values of T_h satisfy $\sigma_n = 1 \quad$ for all n.
5. Conclude that T_h is not compact and does not regularize.

Link between regularity and eigenvalues

Thus $\sigma = 1$ No decrease! **No efficient basis functions for transport and advection-dominated regime**

- No regularity gain
- No compacity
- No decrease in POD eigenvalues
- No efficient basis functions

What can we do in that case?

- shifted POD
- transport maps
- nonlinear reduced bases
- POD + autoencoders

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Link between regularity and eigenvalues

Thus $\sigma = 1$ No decrease!

No regularity gain

No compacity

No decrease in POD eigenvalues

No efficient basis functions

What can we do in that case?

Transport:
$$\begin{cases} \text{shifted POD: } u(x,t) = \sum_k \sum_n a_{k,n}(t)\Phi_{k,n}(\mathbf{x} - \mathbf{s}_k(t)) \\ \text{transport maps: } u(x,t) = \sum_n a_n(t)\Phi_n(T_t(\mathbf{x})) \end{cases}$$

Nonlinear dynamics:
$$\begin{cases} \text{nonlinear reduced bases: } =\text{F(u)} \leftrightarrow \dot{a} = Aa + H(a \otimes a) \\ \text{POD+autoencoders}\ ^1\text{(t,}\mu) \to NN \to a(t,\mu) \to u \simeq Va \end{cases}$$

[1] POD_DL_ROM: https://reducedbasis.github.io/docs/poddlrom/

Sheep model

Let's get back to our sheep

linear second-order parameter dependent problem

$$\begin{cases} -\nabla \cdot \big(a(\boldsymbol{\mu})\nabla u\big) = f(\boldsymbol{\mu}) & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega. \end{cases}$$

$u(\mathbf{x};\boldsymbol{\mu}) \in V$: Unknowns,

$\boldsymbol{\mu} \in \mathcal{G}$: Variable parameter,

How do we construct the reduced basis with the POD in practice?

Snapshot POD

$$\boxed{\mathbb{E}[(u,\Phi)u] = \lambda\Phi, \text{ i. e. } C\Phi = \lambda\Phi}.$$

$\Leftrightarrow \int_\Omega R(\mathbf{x},\mathbf{x}')\ \Phi(\mathbf{x}')d\mathbf{x}' = \lambda\Phi(\mathbf{x})$, with $R(\mathbf{x},\mathbf{x}') = \mathbb{E}[u(\mathbf{x})u(\mathbf{x}')]$, which is the spatial continuous covariance operator.

Snapshot POD

The space of the modes is contained into the snapshots' span.

$$R(\mathbf{x},\mathbf{x}') = \mathbb{E}[u(\mathbf{x})u(\mathbf{x}')] \simeq \frac{1}{N_{train}} \sum_{k=1}^{N_{train}} u(\mathbf{x},\boldsymbol{\mu}_k)u(\mathbf{x}',\boldsymbol{\mu}_k).$$

$$\boxed{\mathbb{E}[\underbrace{(u,\Phi)}_{a}\, u] = \lambda\Phi, \text{ i. e. } C\Phi = \lambda\Phi}.$$

$$\text{so } \frac{1}{N_{train}} \sum_{k=1}^{N_{train}} u(\mathbf{x},\boldsymbol{\mu}_k) \underbrace{\int_\Omega u(\mathbf{x}',\boldsymbol{\mu}_k)\Phi(\mathbf{x}')d\mathbf{x}'}_{a_k=(u(\boldsymbol{\mu}_k),\Phi)} = \lambda\Phi(\mathbf{x}), \qquad \text{(R)}$$

$$\boxed{\text{Therefore } \Phi(\mathbf{x}) = \sum_{k=1}^{N_{train}} \alpha_k u(\mathbf{x},\boldsymbol{\mu}_k).}$$

$$a_k = (u(\boldsymbol{\mu}_k),\Phi) = (u(\boldsymbol{\mu}_k), \sum_{j=1}^{N_{train}} \alpha_j u(\boldsymbol{\mu}_j)) = \sum_{j=1}^{N_{train}} \alpha_j \underbrace{(u(\boldsymbol{\mu}_k), u(\boldsymbol{\mu}_j))}_{\mathbf{C}_{k,j}}$$

Snapshot POD

The space of the modes is contained into the snapshots' span.

$$R(\mathbf{x},\mathbf{x}') = \mathbb{E}[u(\mathbf{x})u(\mathbf{x}')] \simeq \frac{1}{N_{train}} \sum_{k=1}^{N_{train}} u(\mathbf{x},\boldsymbol{\mu}_k)u(\mathbf{x}',\boldsymbol{\mu}_k).$$

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Snapshot POD

$$(R) \Leftrightarrow \frac{1}{N_{train}} \sum_{k=1}^{N_{train}} u(\mathbf{x}, \boldsymbol{\mu}_k) \sum_{j=1}^{N_{train}} \alpha_j \mathbf{C}_{k,j} = \lambda \sum_{k=1}^{N} \alpha_k u(\mathbf{x}, \boldsymbol{\mu}_k)$$

gives for one $k = i$

$$\boxed{\frac{1}{N_{train}} \sum_{j=1}^{N_{train}} \mathbf{C}_{i,j} \alpha_j = \lambda \alpha_i.}$$

Thus, $\mathbf{C}\alpha = N_{train}\lambda\alpha = \lambda'\alpha$: the eigenvalues of the spatial covariance operator or the snapshot correlation matrix are the same (up to a factor N_{train})!

Discretization

$$\mathbf{C}\alpha = \lambda'\alpha \tag{C}$$

We impose (α_n) orthonormal.
Let $S = [u_1, \dots, u_{N_{train}}] \in \mathbb{R}^{\mathcal{N} \times N_{train}}$. Then $\Phi = S\alpha$, and
$\|\Phi\|^2 = (S\alpha, S\alpha) = \alpha^T S^T S\alpha$
But (C) implies that $\|\Phi\|^2 = \lambda'\alpha^T\alpha = \lambda'$

$$\text{Hence } \tilde{\Phi}(\mathbf{x}) = \frac{1}{\sqrt{\lambda'}} \sum_{k=1}^{N_{train}} \alpha_k u(\mathbf{x}, \boldsymbol{\mu}_k) \text{ (after normalization)}$$

Let us denote in the following slides $\Phi = \tilde{\Phi}$ and $\lambda = \lambda'$.

Discretization: Snapshot POD algorithm

1: Collect snapshots $u(\cdot, \mu_i)$, $i = 1, \ldots, N_{train}$
2: Assemble snapshot matrix S
3: Compute correlation matrix $C = S^T S$ or $C = S^T M S$ (M= mass matrix)
4: Solve $C\alpha_i = \lambda_i \alpha_i$, $i = 1, \ldots, N_{train}$
5: Sort the eigenvalues
6: Retrieve first N eigenvalues/eigenvectors
7: Build POD modes $\Phi_i = \frac{1}{\sqrt{\lambda_i}} S\alpha_i$, $i = 1, \ldots, N$

The space of the modes is contained into the snapshots' span: $N \leq N_{train}$
How do we choose N? $\lambda_1 >= \lambda_2 >= \ldots >= 0$

$$\mathbb{E}[\|u - P_N u\|^2] = \sum_{k>N} \mathbb{E}[a_k^2] = \sum_{k>N} \lambda_k$$

Relativ Information Content (RIC) must be close to 0:

$1 - \sum_{k=1}^{N} \lambda_k / \sum_{k=1}^{N_{train}} \lambda_k$

TP VF5-TPFA

Notations

Mesh ($\mathcal{T}$) size: h

Sets of edges: $\mathcal{F}, \mathcal{F}_{ext}, \mathcal{F}_{int}, \mathcal{F}_K$

Normals: $\mathbf{n}_K, \mathbf{n}_{K\sigma}, \mathbf{n}_{KL}$

Volumes / Measures/Distances: $|K|, |\sigma|, d_{K\sigma}, d_{L\sigma}, d_{KL}$

Finite Volume Methods

Based on the conservation form of the PDE

Integrate the balance equation on each cell κ and apply Stokes' formula:

$$\sum_{\text{edges of } \kappa} \text{outward flux} = \int_{\kappa} \text{source.}$$

Approximate each flux and write the discrete balance equation obtained.

TP VF5-TPFA

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Finite Volume Methods

Based on the conservation form of the PDE $\rightarrow$ Flux: total outward flux = the total internal source

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Finite Volume Methods

- Based on the conservation form of the PDE $\rightarrow$ Flux: total outward flux $=$ the total internal source
- Integrate the equation on each cell κ and apply Stokes' formula:

$$\int_K f(\mathbf{x})\, d(\mathbf{x}) = -\int_K \nabla \cdot (a(\boldsymbol{\mu})\nabla u) = \sum_{\sigma \in \mathcal{F}_K} \underbrace{-\int_\sigma a(\boldsymbol{\mu})\nabla u(\mathbf{x}) \cdot \mathbf{n}_{K,\sigma}\ d_\gamma(\mathbf{x})}_{\overline{F}_{K,\sigma}}$$

Approximate each flux and write the discrete balance equation obtained.

TP VF5-TPFA

Notations

Mesh ($\mathcal{T}$) size: h

Sets of edges: $\mathcal{F}, \mathcal{F}_{ext}, \mathcal{F}_{int}, \mathcal{F}_K$

Normals: $\mathbf{n}_K, \mathbf{n}_{K\sigma}, \mathbf{n}_{KL}$

Volumes / Measures/Distances: $|K|, |\sigma|, d_{K\sigma}, d_{L\sigma}, d_{KL}$

Flux balance:

$$\sum_{\sigma \in \mathcal{F}_K} \overline{F}_{K,\sigma} = \int_K f(\mathbf{x})\ d(\mathbf{x}).$$

Flux conservativity:

$$\overline{F}_{K,\sigma} + \overline{F}_{L,\sigma} = 0 \text{ if } \sigma = K|L.$$

We want to find $u_h = (u_K)_{K \in \mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$
Define $u_h \to F_{K,\sigma}(u_h)$ that approximates the flux and find $u_h \in \mathbb{R}^{\mathcal{T}}$ such that
$|K| f_K = \sum_{\sigma \in \mathcal{F}_K} F_{K,\sigma} \forall K \in \mathcal{T}$.

TP VF5-TPFA

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.

TP VF5-TPFA

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.

Case of an interior edge $\sigma \in \mathcal{E}_{int}, \ \sigma = K|L$

$$x_L - x_K = d_{KL}\, \mathbf{n}_{KL}.$$

If $x \in \sigma$,

$$(\nabla u(x)) \cdot \mathbf{n}_{KL} = \frac{u(x_L) - u(x_K)}{d_{KL}} + \mathcal{O}(h).$$

$$\Longrightarrow \overline{F}_{K,\sigma} = \underbrace{-\overline{A(\boldsymbol{\mu})}|\sigma| \frac{u(x_L) - u(x_K)}{d_{KL}}}_{F_{K,\sigma}(u_h)} + \mathcal{O}(h^2)$$

where $\overline{A}$ is the harmonic average: $\overline{A} = \frac{A(x_L)A(x_K)d_{KL}}{A(x_L)d_{K,\sigma}+A(x_K)d_{L,\sigma}}$

TP VF5-TPFA

We want to find $u_h = (u_K)_{K \in \mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$
Define $u_h \to F_{K,\sigma}(u_h)$ that approximates the flux and find $u \in \mathbb{R}^{\mathcal{T}}$ such that

$$|K| f_K = \sum_{\sigma \in \mathcal{F}_K} F_{K,\sigma}, \ \forall K \in \mathcal{T}$$

.

Case of a boundary edge $\sigma \in \mathcal{E}_{ext}$

$$x_\sigma - x_K = d_{K\sigma} \, \mathbf{n}_{K\sigma}.$$

$$(\nabla u(x)) \cdot \mathbf{n}_{K\sigma} \approx \frac{u(x_\sigma) - u(x_K)}{d_{K\sigma}} = \frac{0 - u(x_K)}{d_{K\sigma}} \qquad \text{(boundary condition)}$$

$$\Longrightarrow \overline{F}_{K,\sigma} = \underbrace{-|\sigma| A_K \frac{-u(x_K)}{d_{K\sigma}}}_{F_{K,\sigma}(u_h)} + \mathcal{O}(h^2)$$

TP VF5-TPFA

Find $u_h = (u_K)_{K \in \mathcal{T}_h}$ such that for all K in $\mathcal{T}_h$:

$$\sum_{\sigma \in F_K \cap F_{int}} \tau_\sigma (u_K - u_L) + \sum_{\sigma \in F_K \cap F_{ext}} \tau_\sigma u_K = \int_K f(x) dx$$

with Dirichlet boundary $u = 0$ on $\partial\Omega$, where $\tau_\sigma = |\sigma| \frac{A_K \ A_L}{A_L \ d_{K,\sigma} + A_K \ d_{L,\sigma}}$ on F_{int} and $\tau_\sigma = |\sigma| \frac{A_K}{d_{K,\sigma}}$ on F_{ext}

We take here $\Omega = [0,1] \times [0,1]$ with a cartesian mesh.
$A(x, y; \mu) = 2\mu_1 + \mu_2 \ sin(x + y) \ cos(xy)$
$f(x, y; \mu) = \mu_3(1 - y) + \mu_4 x \ (1 - x)$

TP VF5-TPFA

POD-based Reduced Order Model with TPFA

- Complete the function `assemble_tpfa`. The TPFA solver must return the cell centers, the matrix M, and the vector b such that $Mu = b$.
- Generate a training dataset:

  Use $N_{train} = 10$ snapshots and sample random parameters $\boldsymbol{\mu}$ with components in $[0, 1]$.
  Solve the full-order TPFA system for each sampled parameter.
  Store the resulting solutions as a snapshots list.
- Using the discrete L^2 inner product $(u, v)_{L^2} = \sum_K |K| \, u_K v_K$,

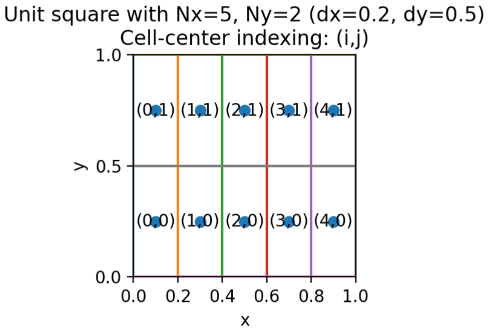
  Assemble the snapshot correlation matrix.
  Compute the reduced basis with a Proper Orthogonal Decomposition (POD).
  Verify that the reduced basis is orthonormal with respect to $(\cdot, \cdot)_{L^2}$.
- Determine how many modes N are sufficient using the Relative Information Content.
- Write a function that computes the ROM projection coefficients of a given full-order solution u.
- Consider a new parameter $\boldsymbol{\mu}$. Write a function that computes the reduced-order approximation of the solution without computing the HF solution.
- Show that the obtained reduced system has the form $\widetilde{M}a = \widetilde{b}$, where $\widetilde{M}$ is of size $N \times N$ and $\widetilde{b}$ is of size N.

TP VF5-TPFA

POD-based Reduced Order Model with TPFA

- Test the reduced model for $\boldsymbol{\mu} = (0.6, 0.5, 0.2, 0.8)$.
- Compare the errors $\|u_{\mathrm{ref}} - u\|_{L^2}$ and $\|u_{\mathrm{ref}} - u_N\|_{L^2}$, where u_{ref} is a refined solution.

TP


Unit square with Nx=5, Ny=2 (dx=0.2, dy=0.5)
Cell-center indexing: (i,j)
(0,1)
(1,1)
(2,1)
(3,1)
(4,1)
(0,0)
(1,0)
(2,0)
(3,0)
(4,0)
y
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
x

TP: FEM

with Kolmogorov n width not small: $u(x,\mu) = tanh(\frac{x-\mu}{\delta})$.

TP: FEM

FEM with Kolmogorov n width not small: Burgers equation:
Viscous Burgers (1D) with periodic BC using scikit-fem

$$u_t + \nu u \times u_x - \varepsilon u_x x = 0 \text{ in } (0, T]x[-1, 1]$$
$$u(0, x) = u0(x) = lam + sin(x)$$
$$u \text{ periodic at } x = -1 and x = 1$$

Time stepping: IMEX (explicit convection, implicit diffusion)
$(M + dt\ \varepsilon\ K)u^{n+1} = Mu^n - dt\ F(u^n)$ where $F_i(u^n) = \int \nu v_i\ (u^n)\ (u^n)_x\ dx$