Mathematics and Art: Exploring connections

Elise Grosjean

Art is a diverse range of human activity and that makes people react

- **Senses**
- **Emotions**
- **Creativity**

...

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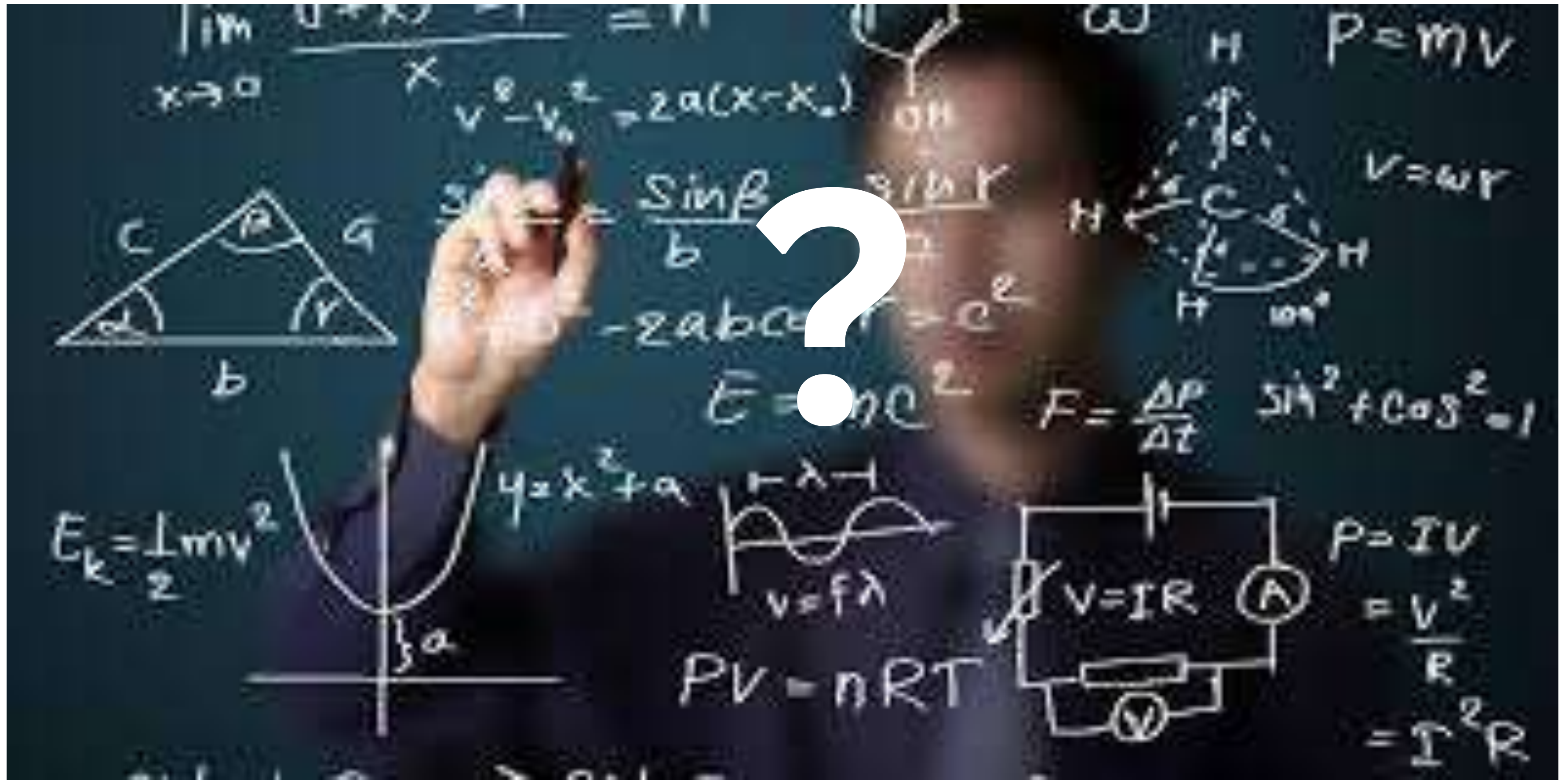

Mathematics

Current mathematics:

- **Axioms**: we start with a small number of statements, assumed to be a priori true
- **Proofs**: what is an implication, an equivalence, ...
- **Theorems, lemma, corollaries**

From the axioms, we therefore obtain theorems which gradually enrich mathematical theory. Because of the unproven bases (the axioms), the notion of "truth" of mathematics is subject to debate.

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A famous syllogism

Everything that is rare is expensive

A cheap horse is a rare thing

therefore a cheap horse is expensive

Mathematics

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EXAMPLE OF AXIOMS USED FOR TILING

Euclid' axioms

[The parallel postulate] Given a line and a point not on it, at most one line parallel to the given line can be drawn through the point

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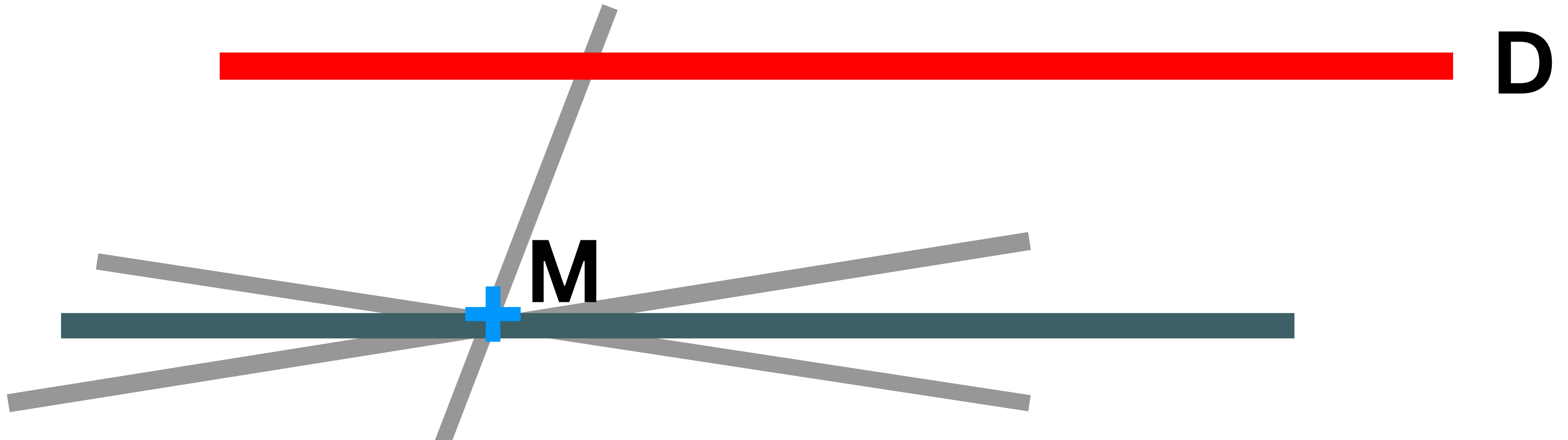

TILING
Link with mathematics

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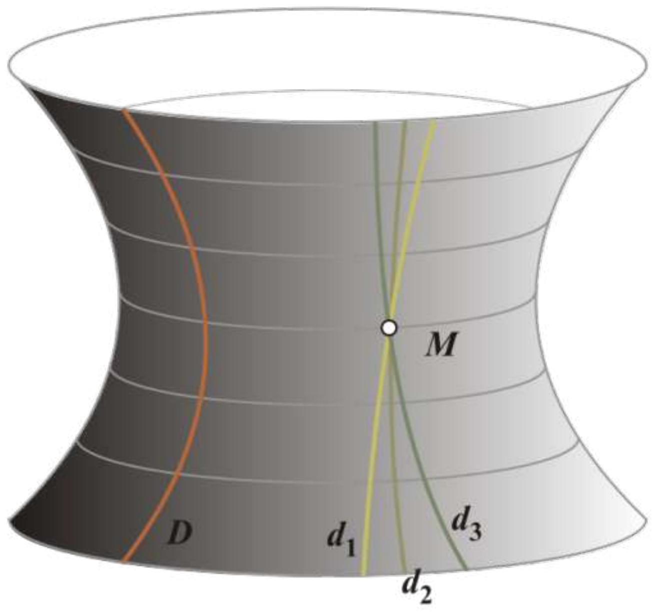


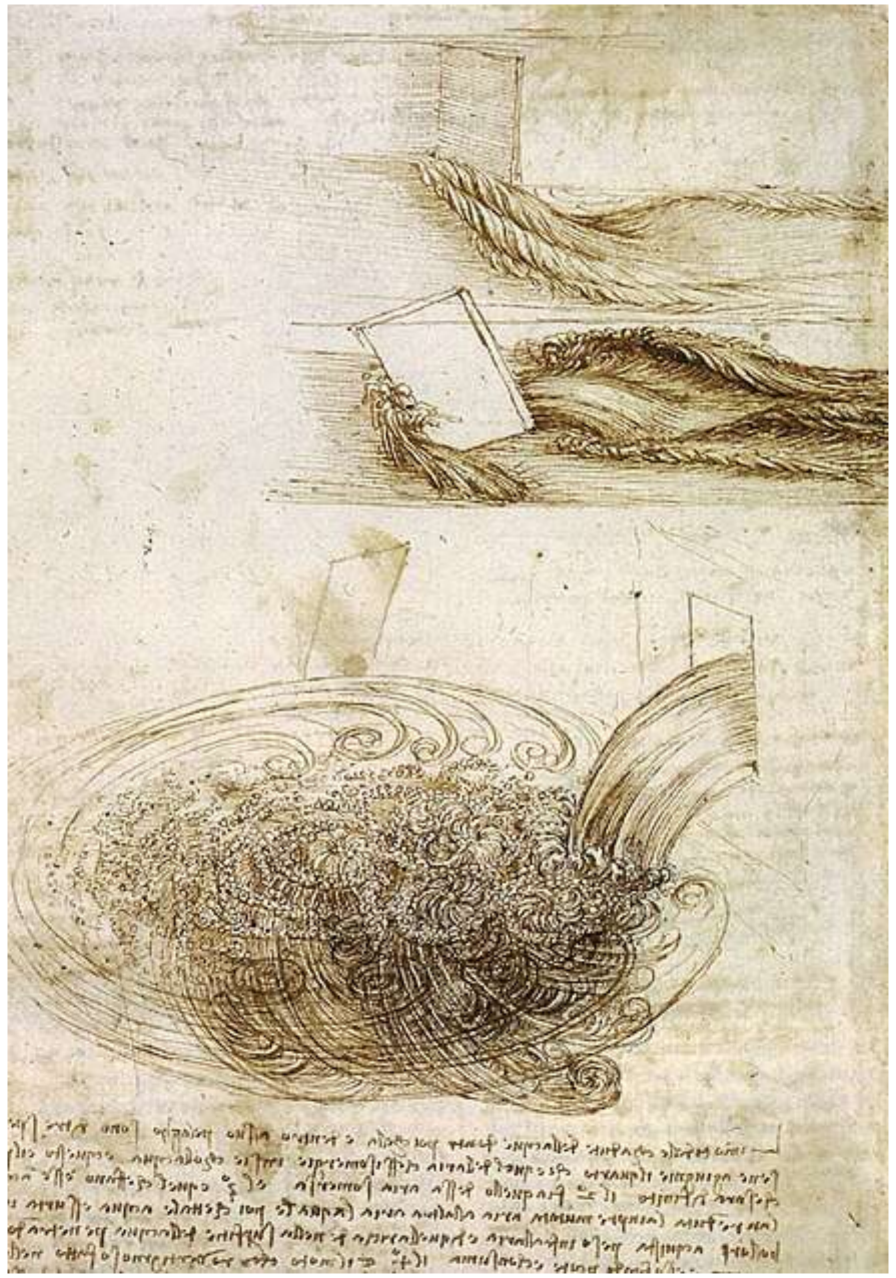
ART IN MATHEMATICS

A prime number is a natural number that can only be divided by itself and by one

$$x \in \mathbb{N}^*, \forall y, \forall z,$$

$$(y \cdot z = x) \rightarrow (y = 1 \vee z = 1)$$

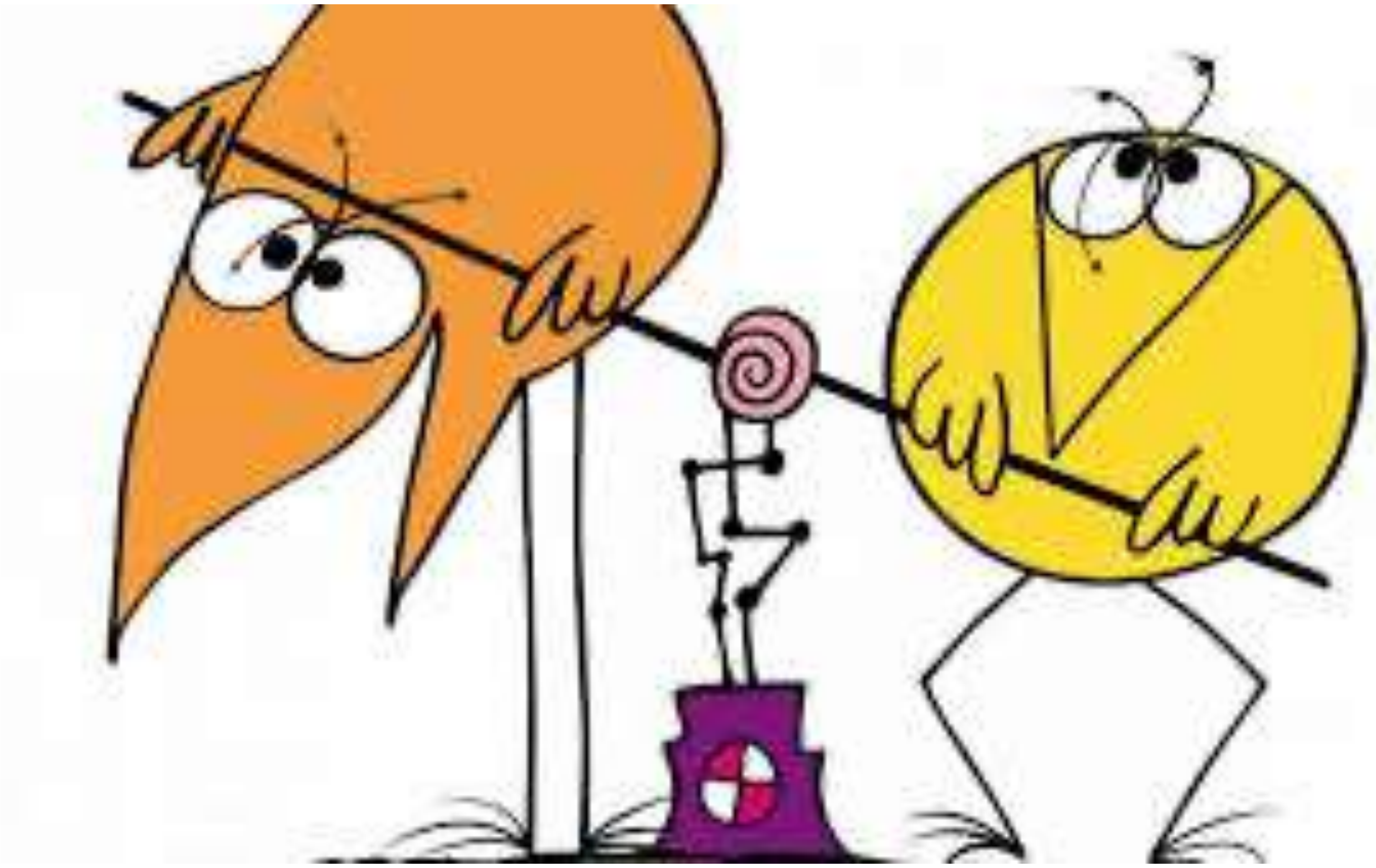
ART IN MATHEMATICS

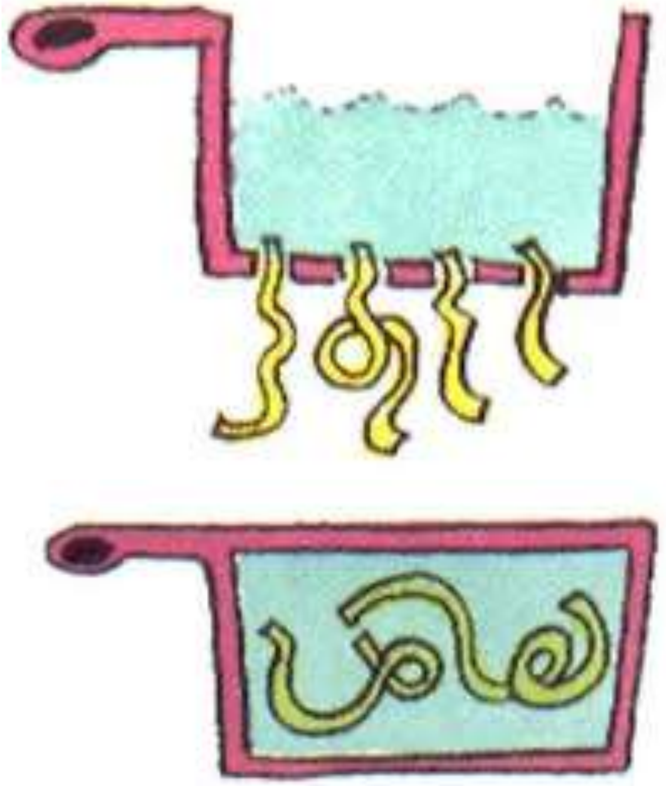


$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\ \rho g_x$$

ART IN MATHEMATICS

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$





SHADOKS THEOREM
The notion of sieve is independent of the notion of holes, and vice versa

ART IN MATHEMATICS

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Mathematics and Art:

Exploring connections

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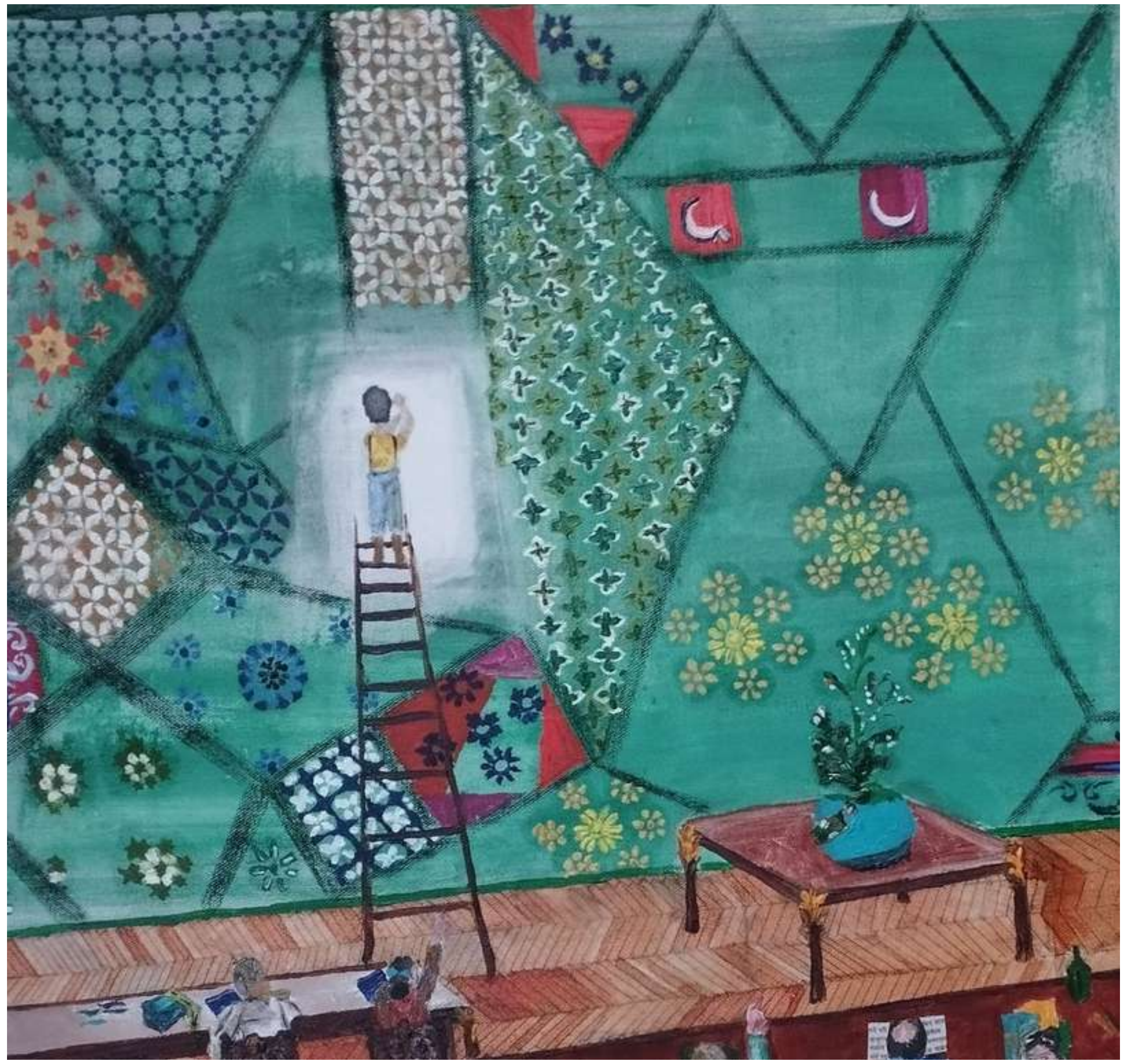
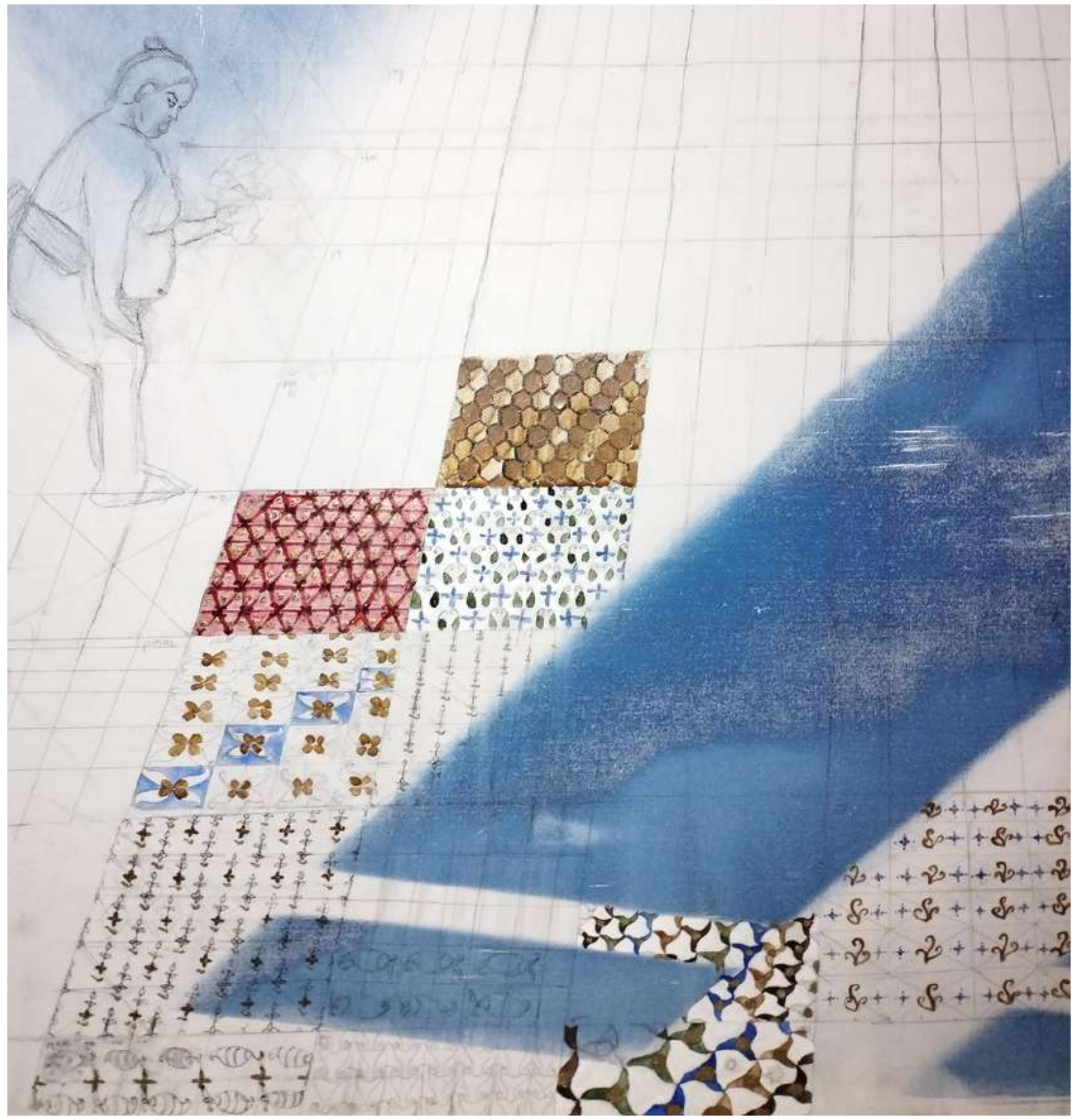
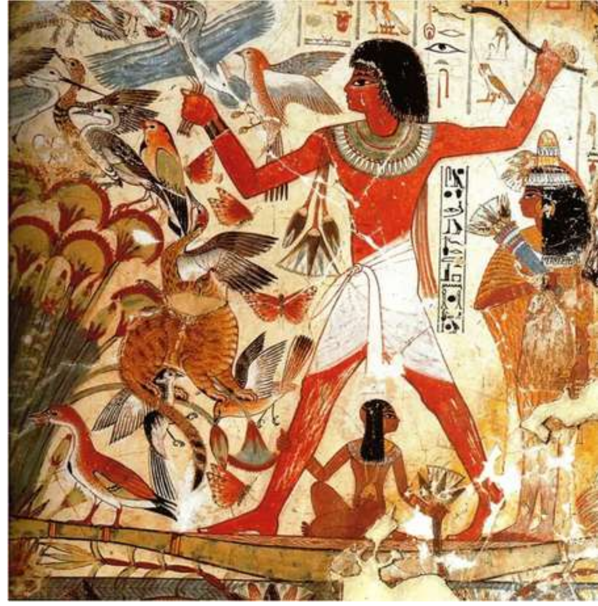
Elise Grosjean





01

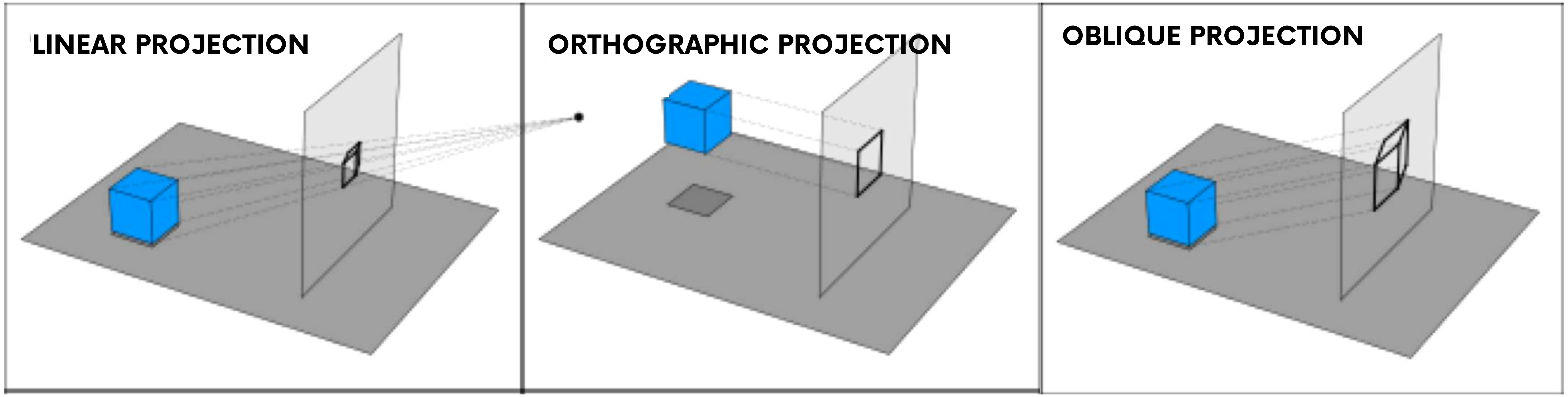
GEOMETRY AND INFINITY



01 GEOMETRY

Several kind of perpectives

- Vertical perspective (Art of Ancient Egypt)
- Parallel projection (Oblique, orthographic)
- Linear projection (Vanishing points)



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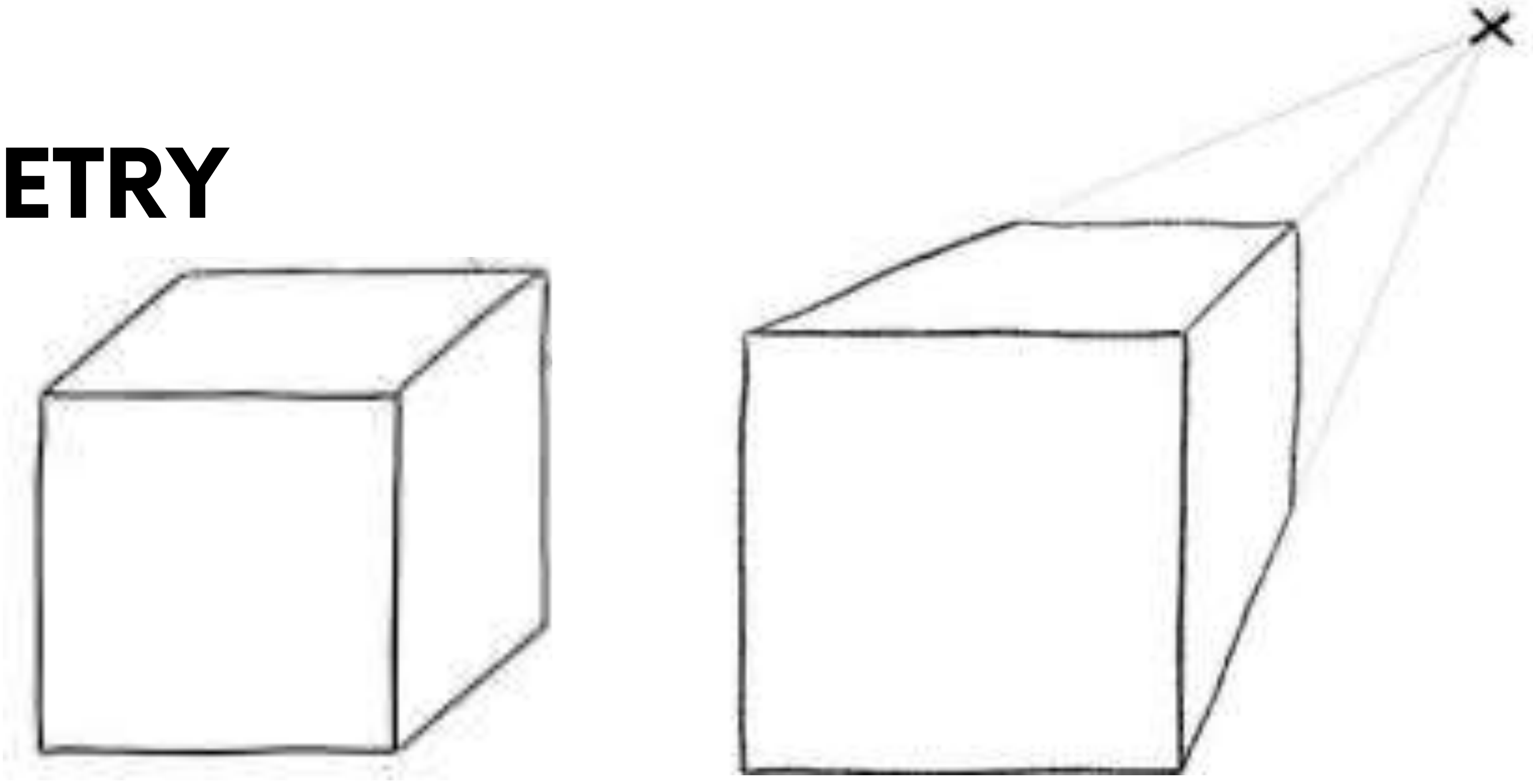

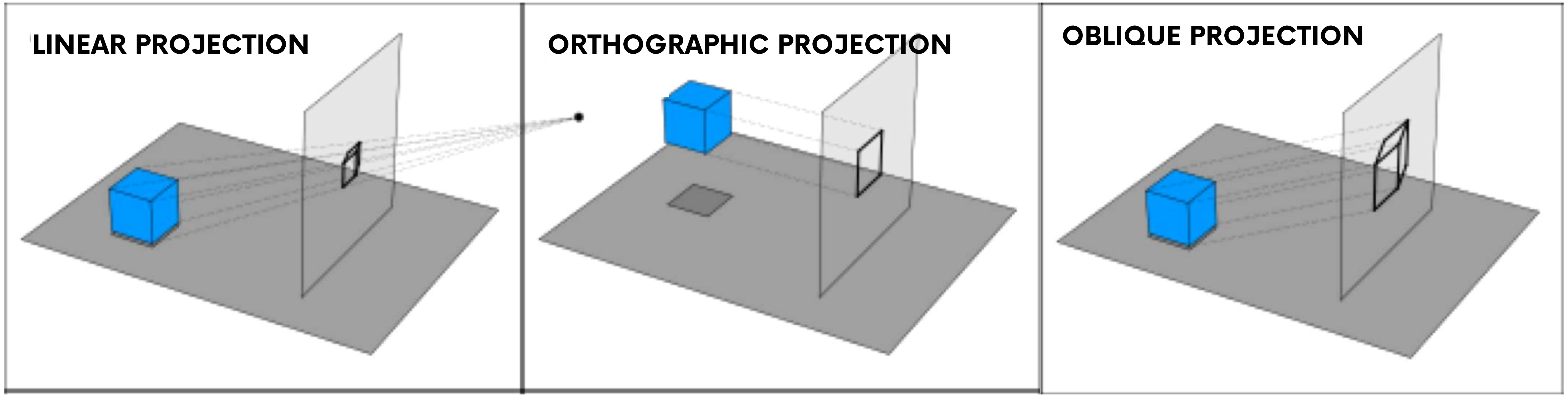

GEOMETRY

Several kind of perpectives

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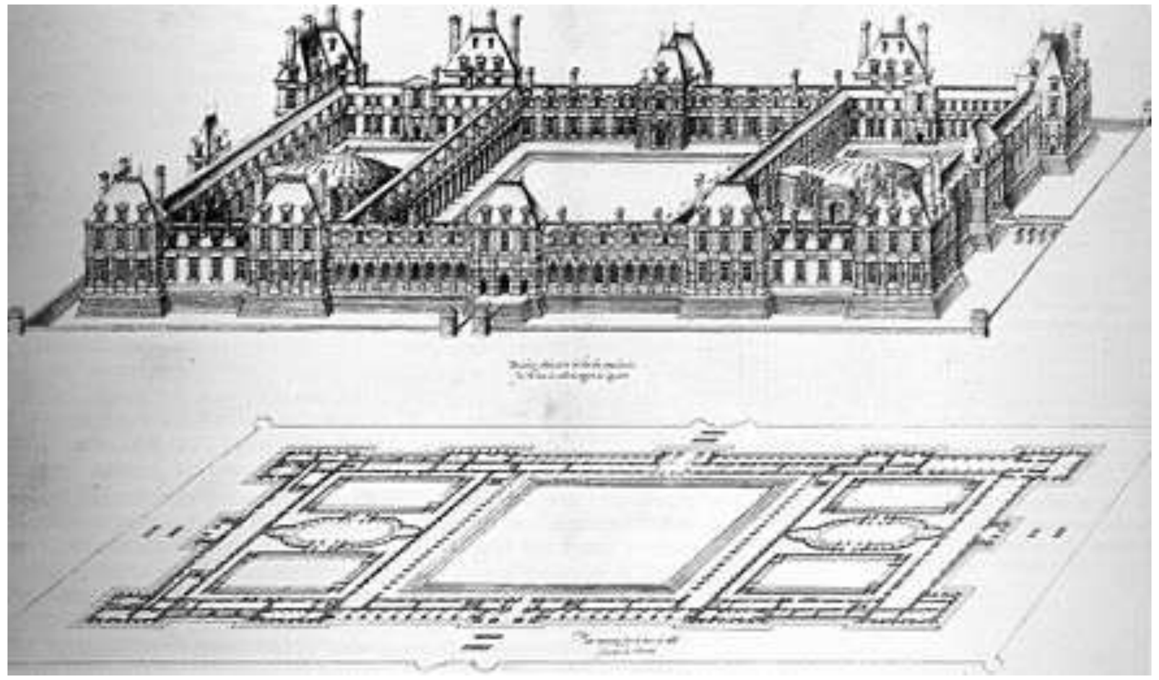
- Architecture
- Chinese art (from 1-2nd until the 18th C.)
- Japanese painting as in Ukiyo-e

First studies of vanishing points around the 14th C.
- Brothers Lorenzetti
- Leon Battista Alberti
- Filippo Brunelleschi
- Raffaello Sanzio

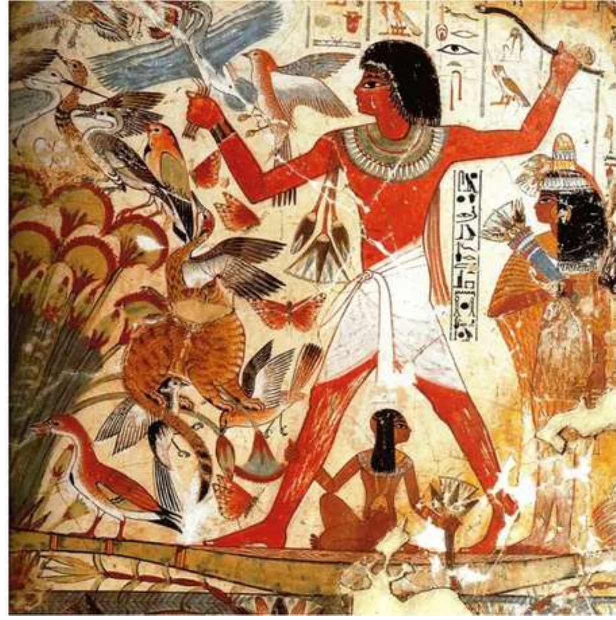
Ambrogio Lorenzetti (1344)

Katsushika Hokusai (1832)
The kannon of the pure Waterfall at Sakanoshita on the Tokaido Road



Jacques 1er Androuet du cerceau (16th C.)
Cavalier perspective of The Tuileries

01 GEOMETRY



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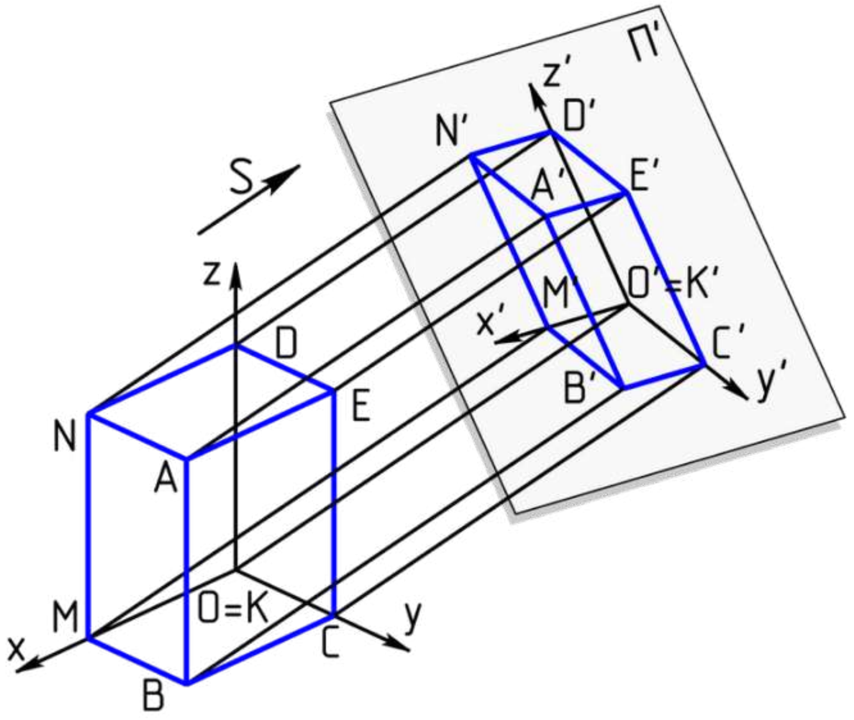
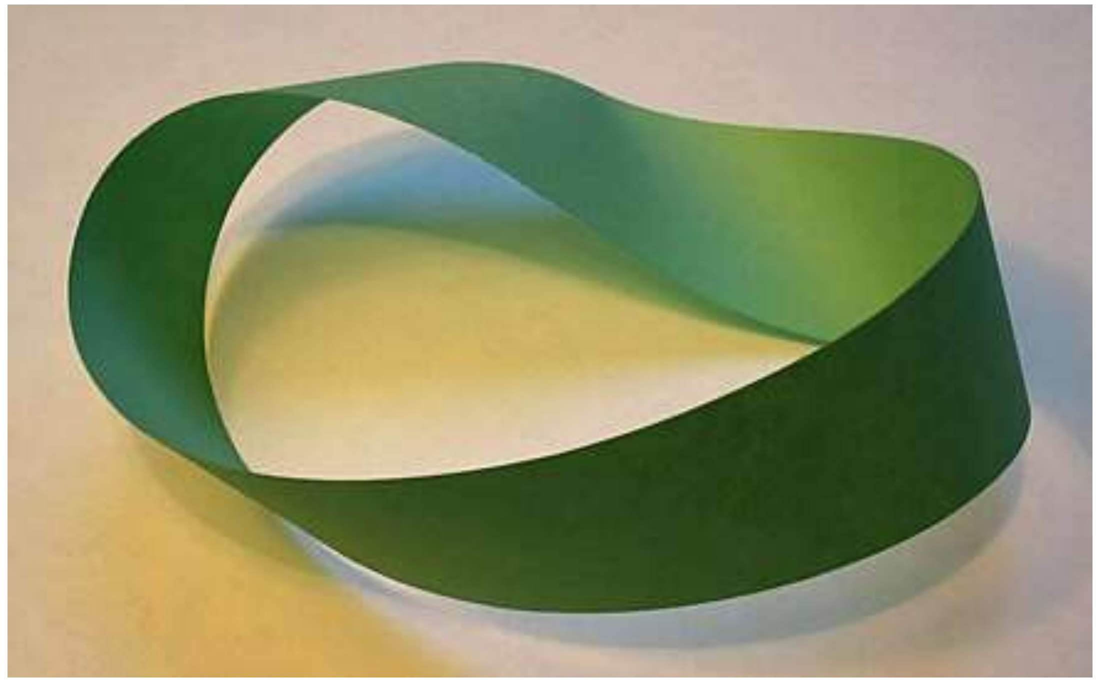


Geometry can be studied analytically:
One can study curves, nodes, strange forms like möbius strip.

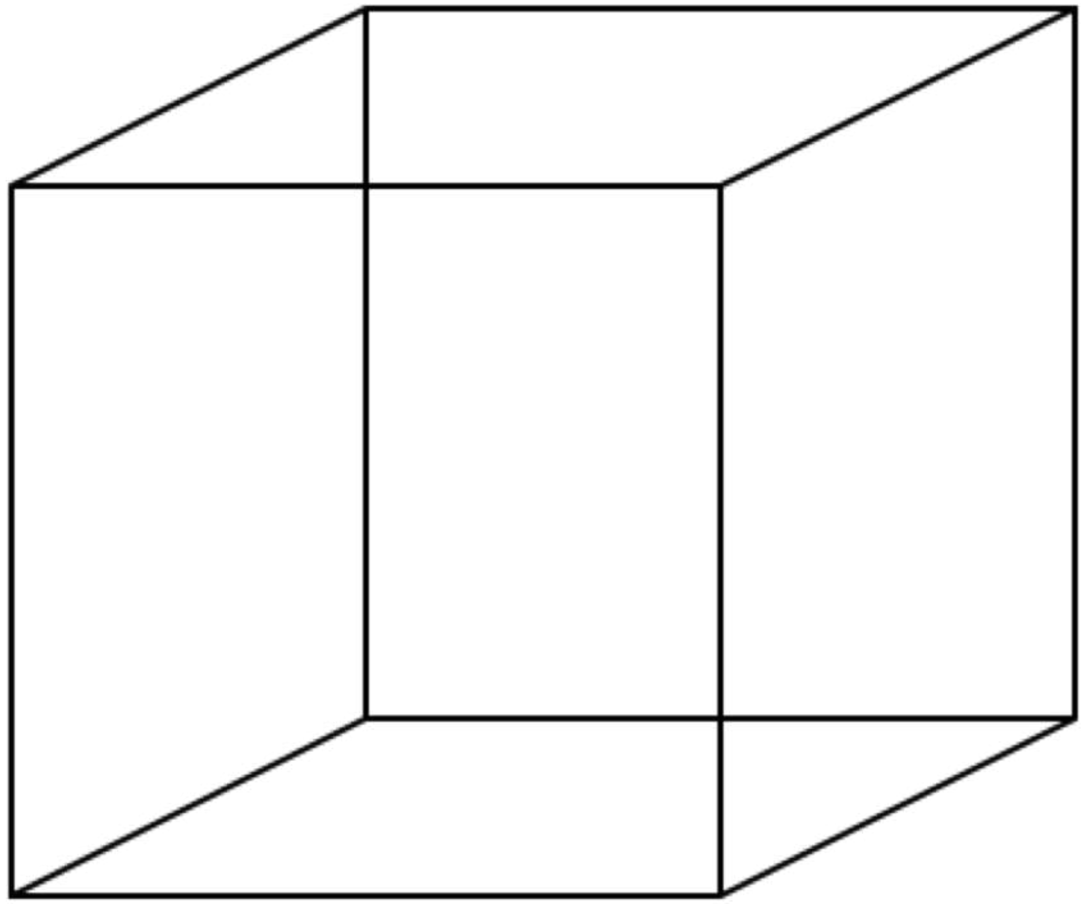


$$\begin{cases} x = \left(1+\frac{t}{2}\cos\frac{v}{2}\right)\cos v \\ y = \left(1+\frac{t}{2}\cos\frac{v}{2}\right)\sin v \\ z = \frac{t}{2}\sin\frac{v}{2} \end{cases} \qquad \begin{matrix} -1 \le t \le 1 \\ 0 < v \le 2\pi \end{matrix}$$

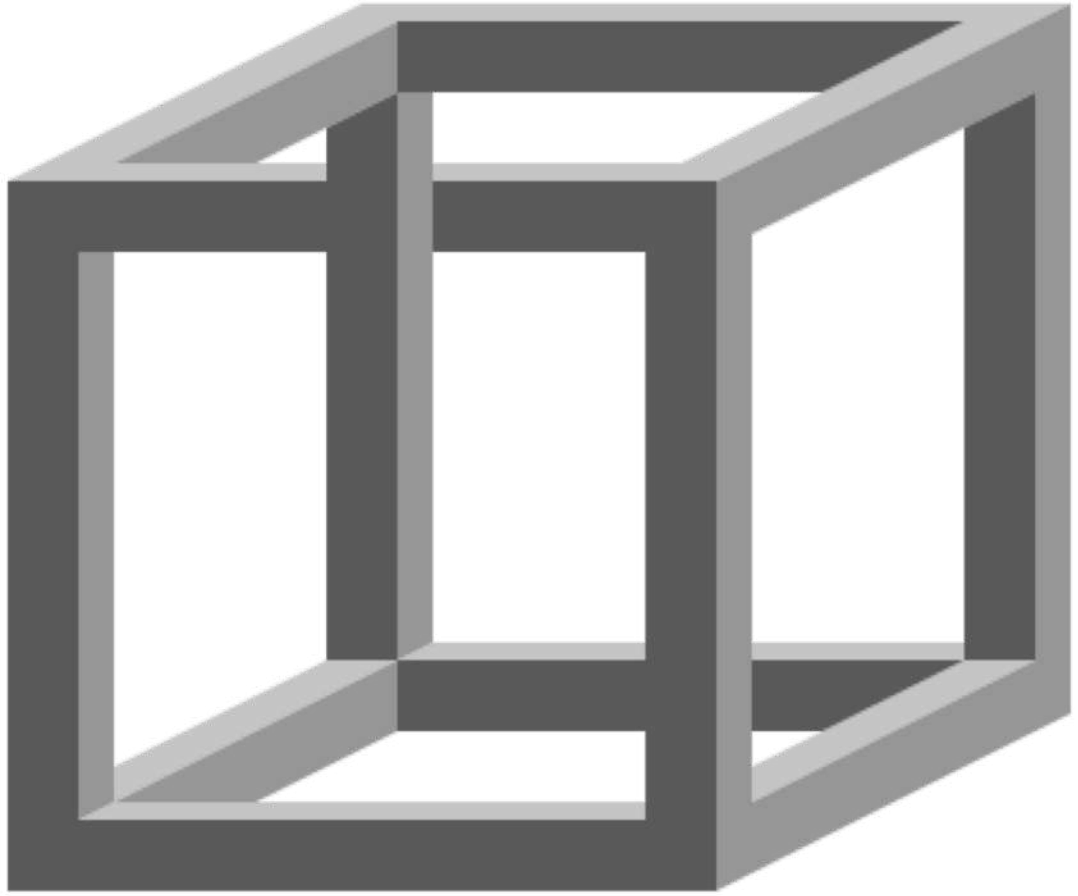
PLAYING WITH PERSPECTIVE

ROGER PENROSE

MAURITS CORNELIS ESCHER.

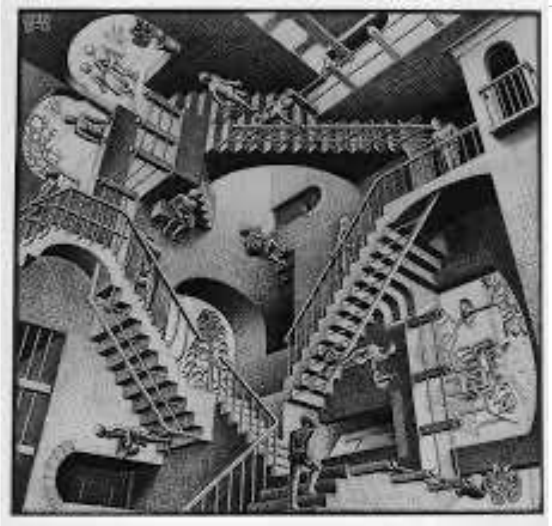


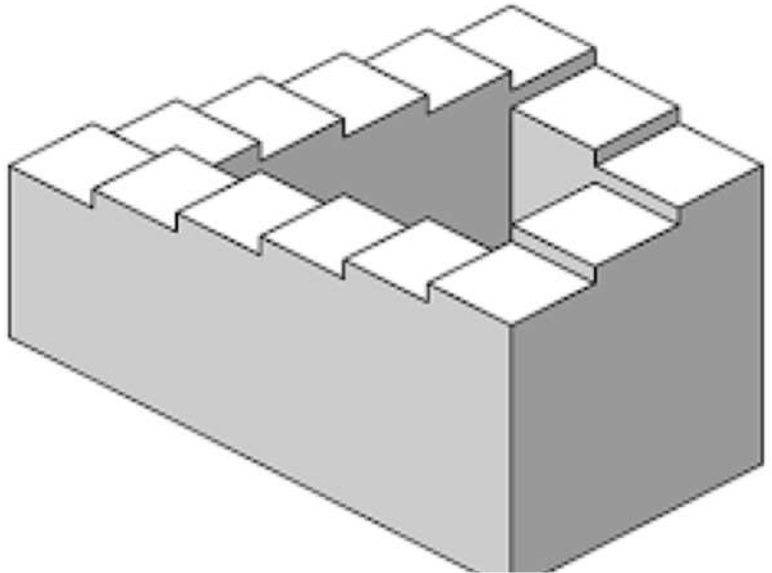
Necker cube



Impossible cube

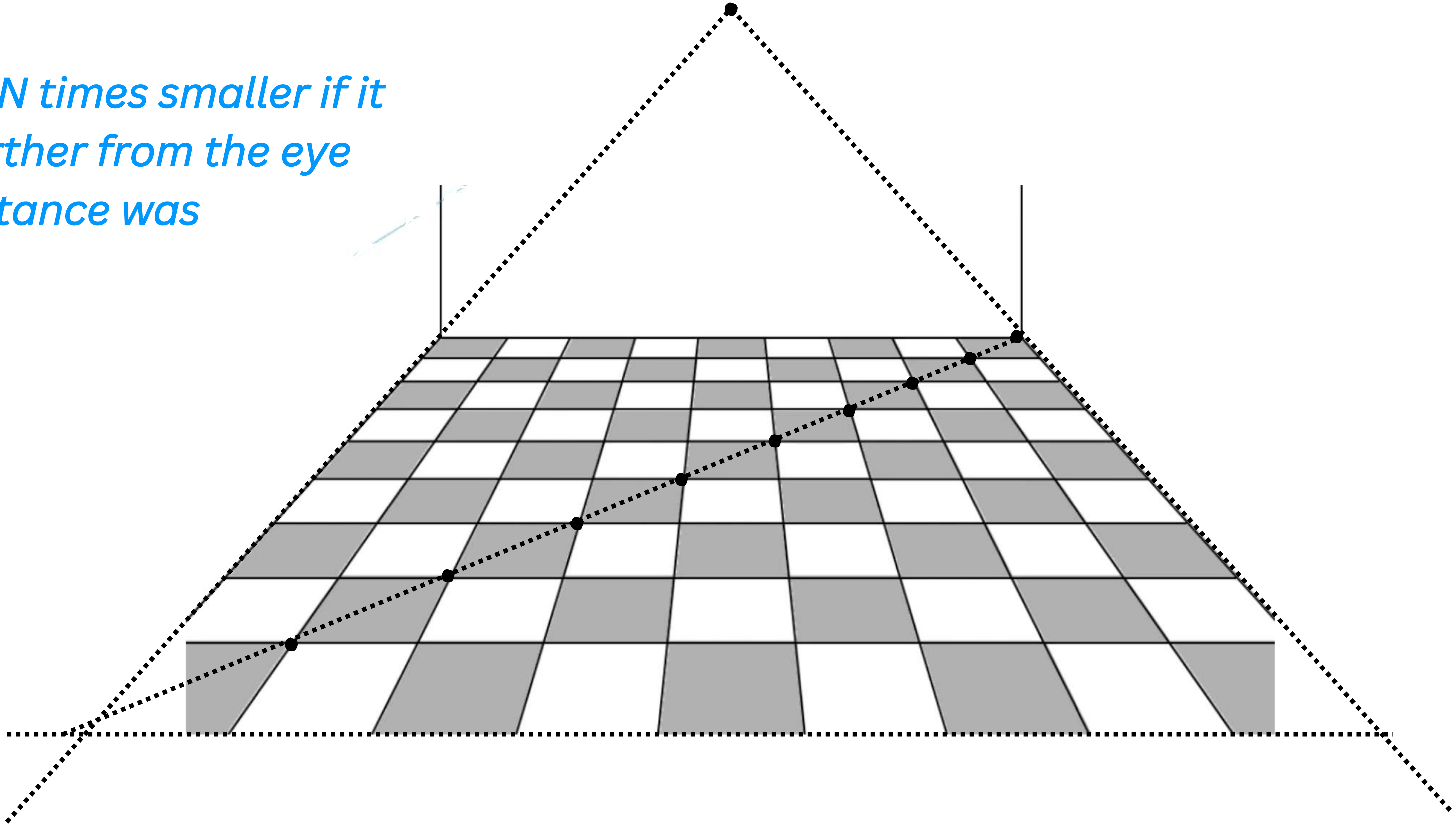
Penrose triangle





A theorem for one vanishing point:
Thales's theorem

For a person, an object looks N times smaller if it has been moved N times further from the eye than the original distance was



A theorem for one vanishing point:
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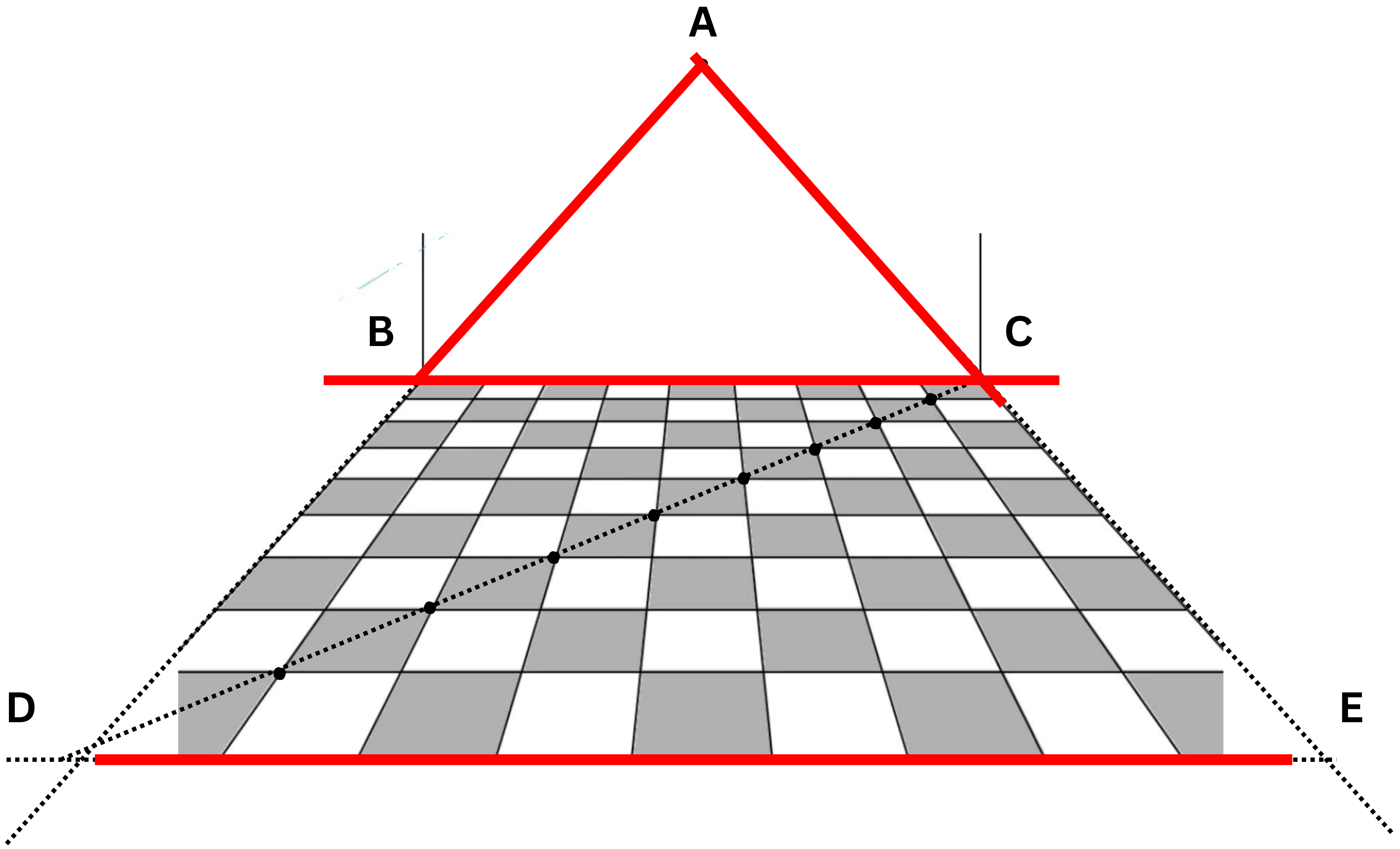
$$\frac{AB}{AD} = \frac{AC}{AE} = \frac{BC}{DE}$$



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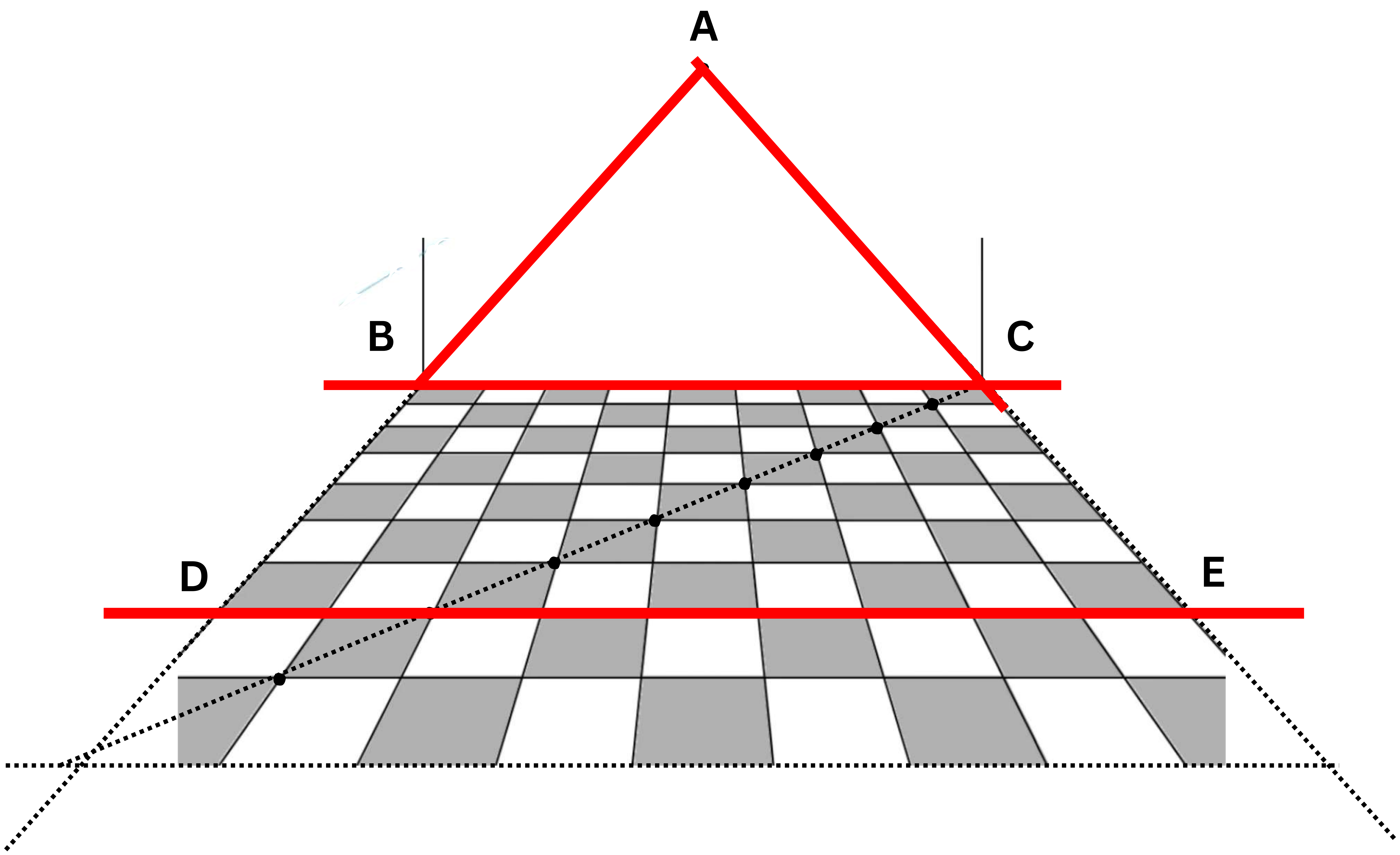

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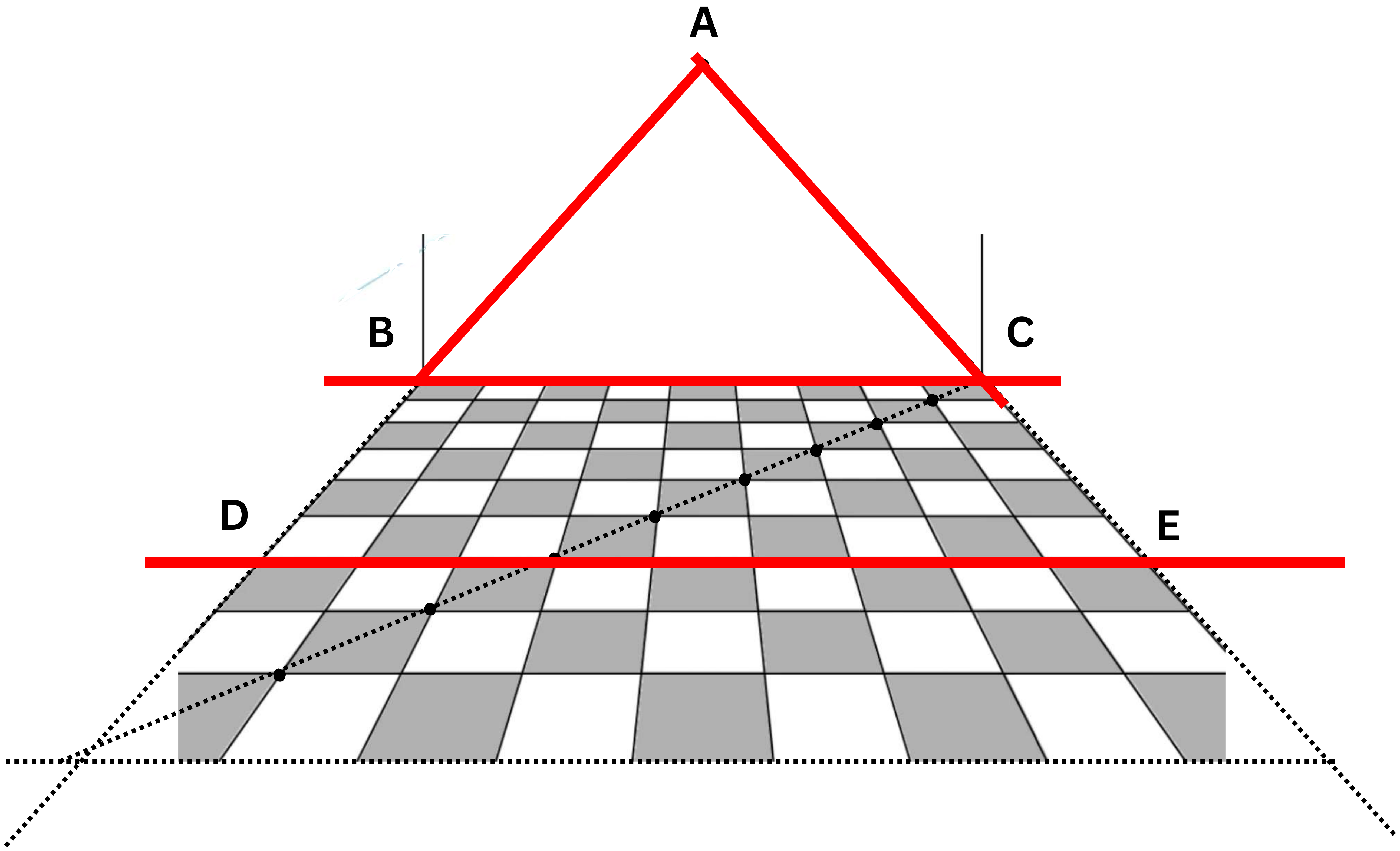
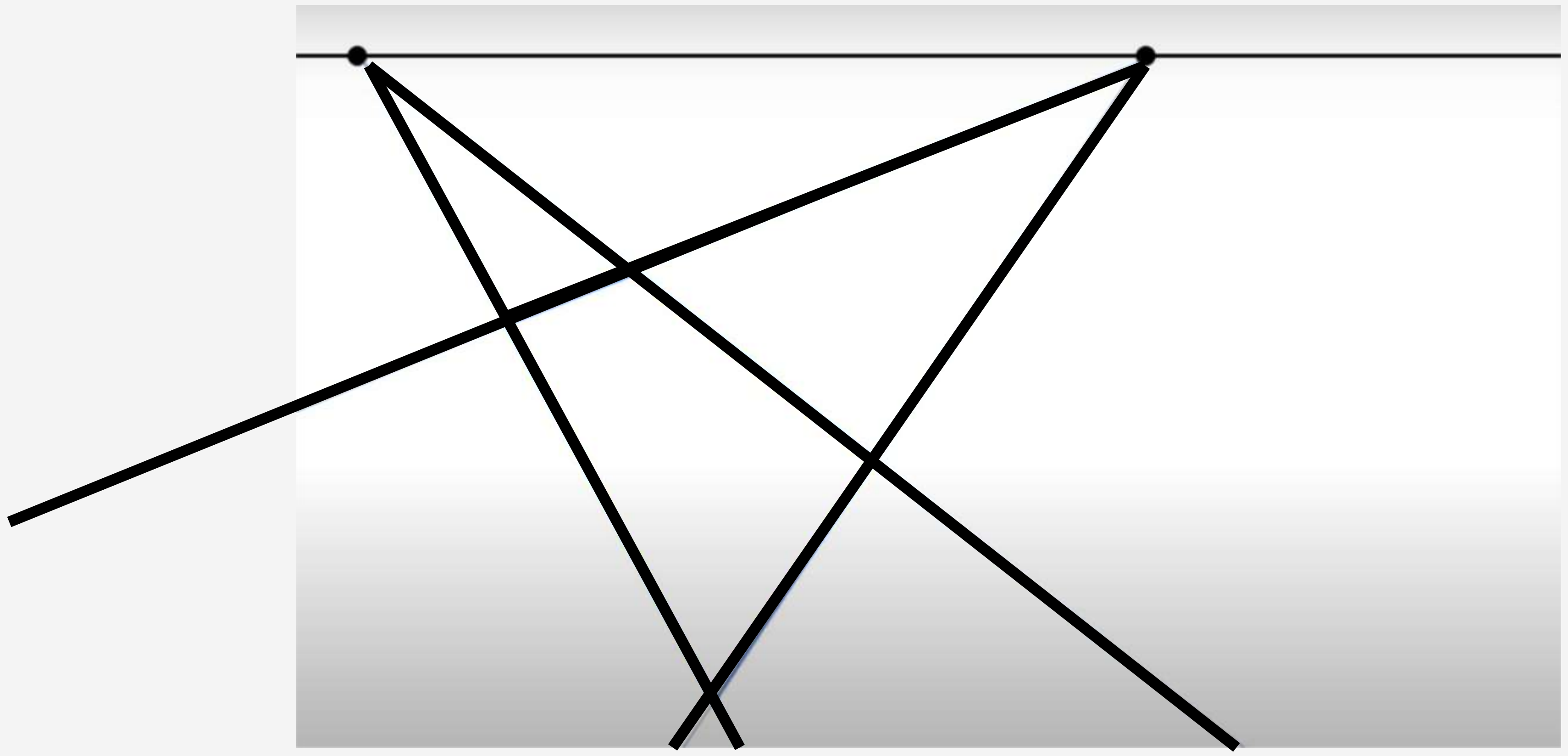
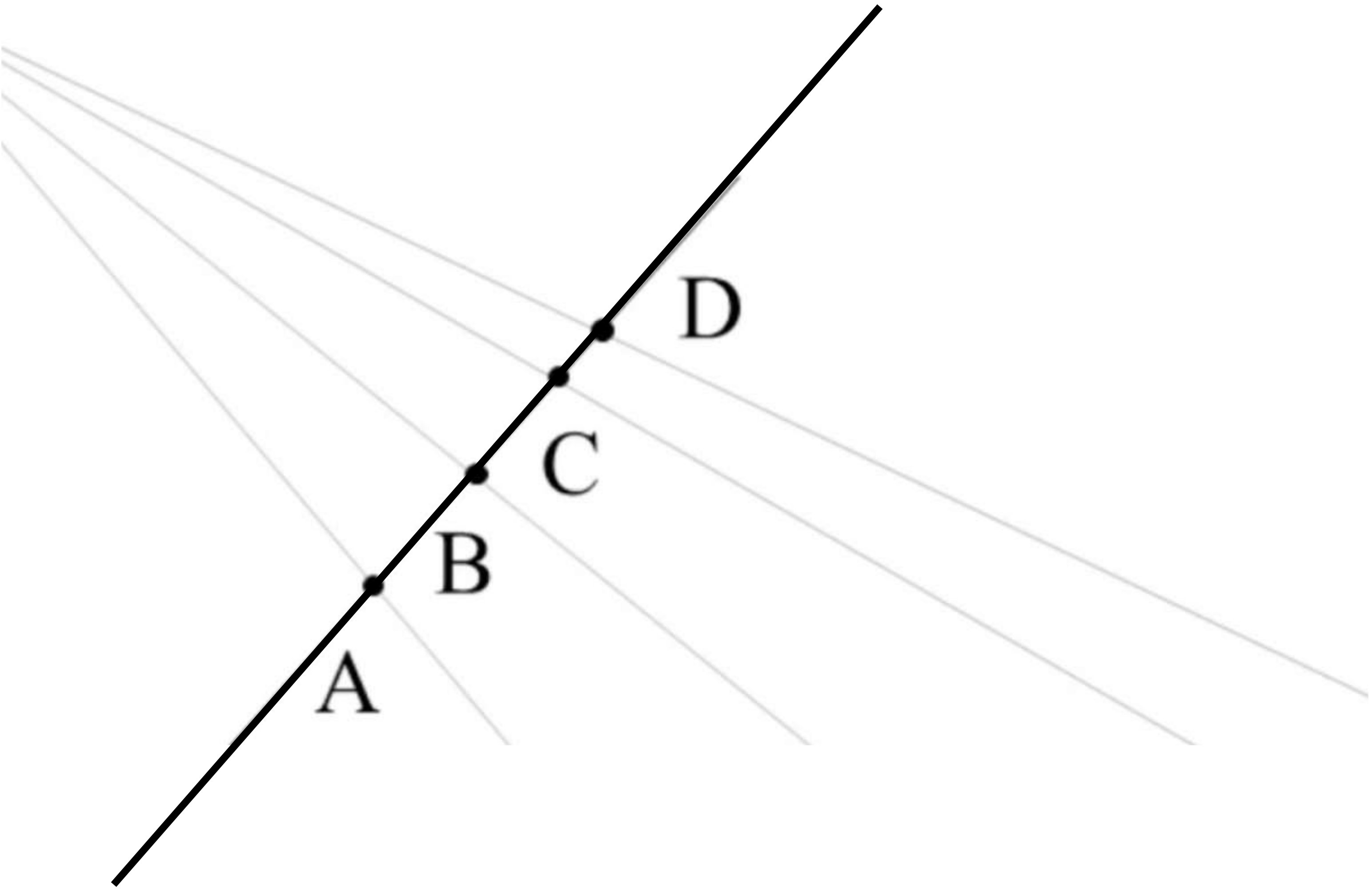


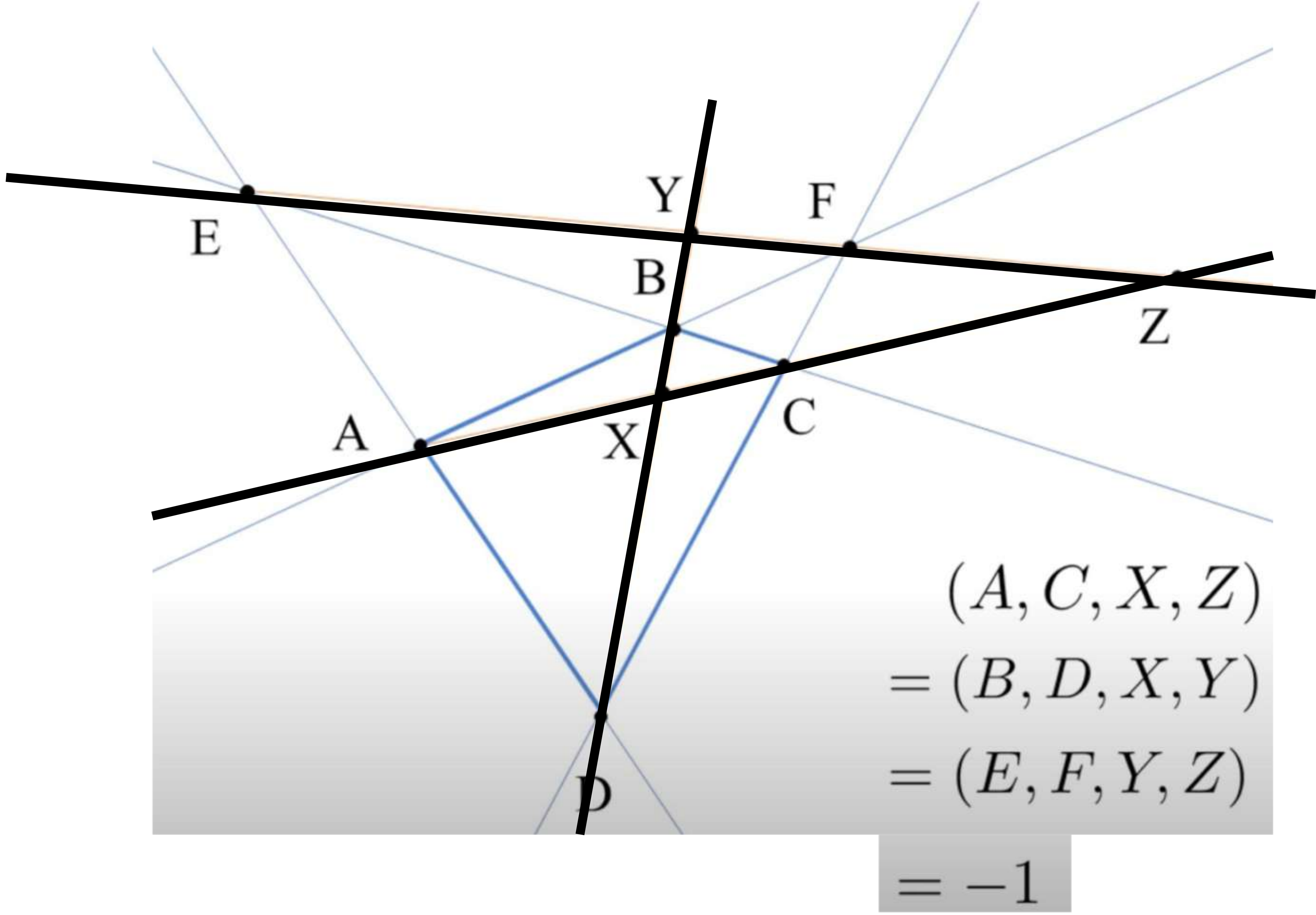
A theorem for two vanishing points:
Complete quadrilateral theorem



A theorem for two vanishing points: Complete quadrilateral theorem

$$(A, B, C, D) = \frac{AC \times BD}{BC \times AD}$$





INFINITY $\aleph$

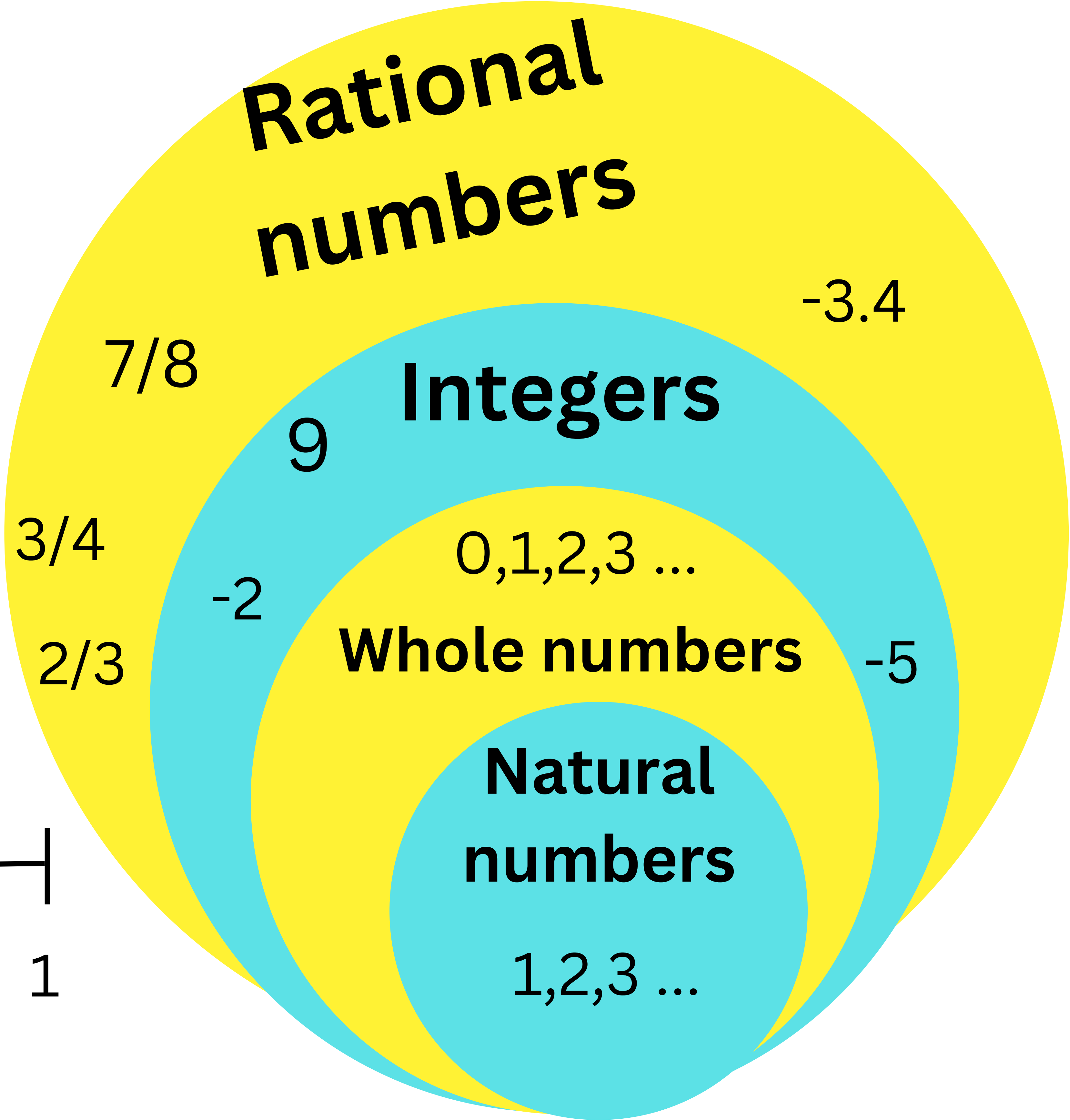
- **Different kinds of infinity (some are bigger than others)**

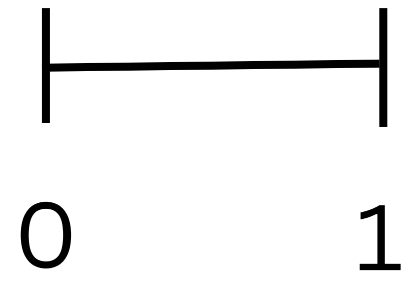
GEORG CANTOR

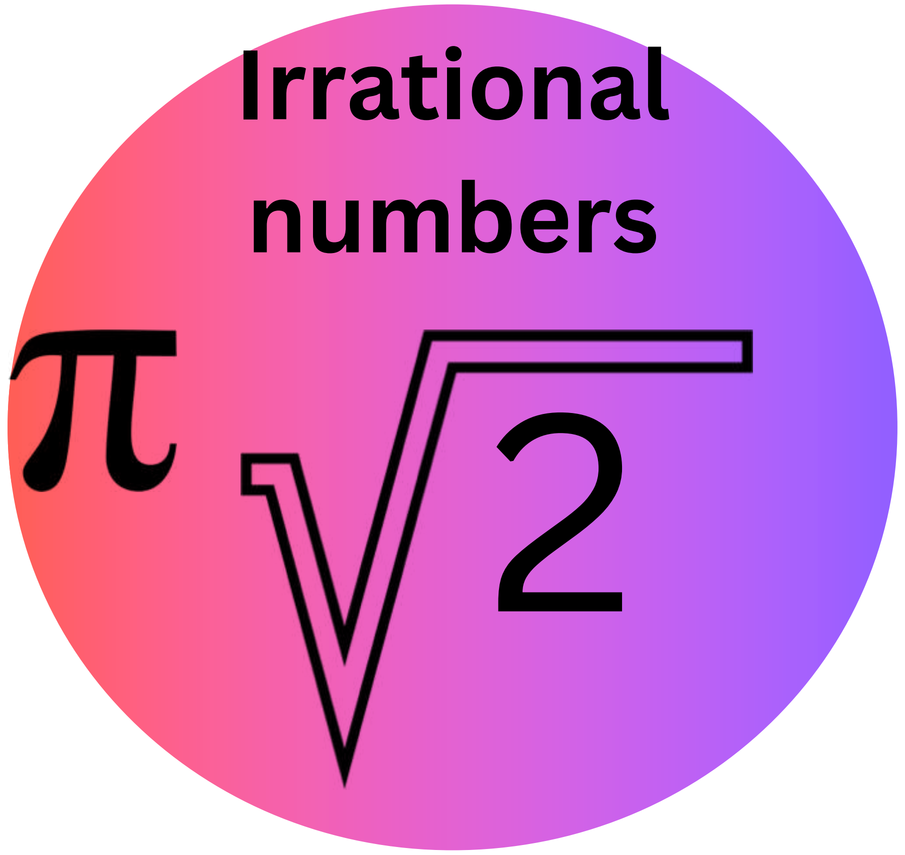
In mathematics (in set theory) the numbers $\aleph$ are a sequence of numbers used to represent the size of infinite sets that can be well-ordered.

ex : $\aleph_0$ represents the size of the natural numbers {0, 1, 2, 3, ... }.

$\aleph_1$ is the size of the infinities of infinite sets = Real numbers cardinality (Cantor continuum hypothesis)







INFINITY $\aleph$

SMALL EXERCISE TO PROVE THAT 0,99999... = 1

- **Different kinds of infinity (some are bigger than others)**

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INFINITY

$\aleph$

∞

REPEATED PATTERNS

- Labyrinth (Chartres Cathedral)
- Tiling (Escher' patterns)
- Fractals (Mandelbrot)
- Mise en abyme (Face of war, Salvador Dali)

PERPETUAL MOVEMENT

- Pendulum

Jose de Rivera (1967, Washington)
Infinity

John Wallis (1616-1703, English mathematician):
J. Wallis popularised the symbol ∞ for infinity.

- **Different kinds of infinity (some are bigger than others)**

GEORG CANTOR

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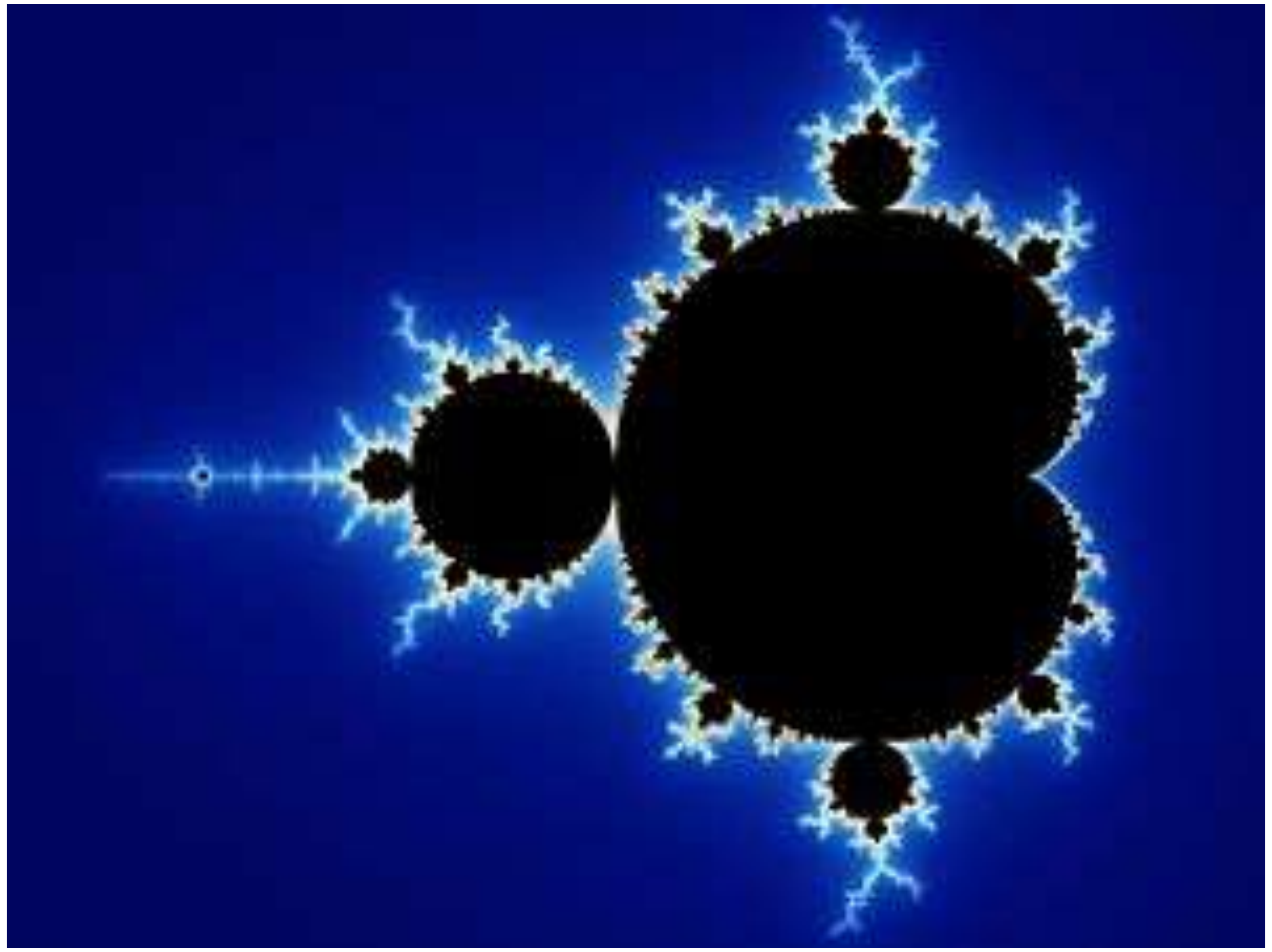
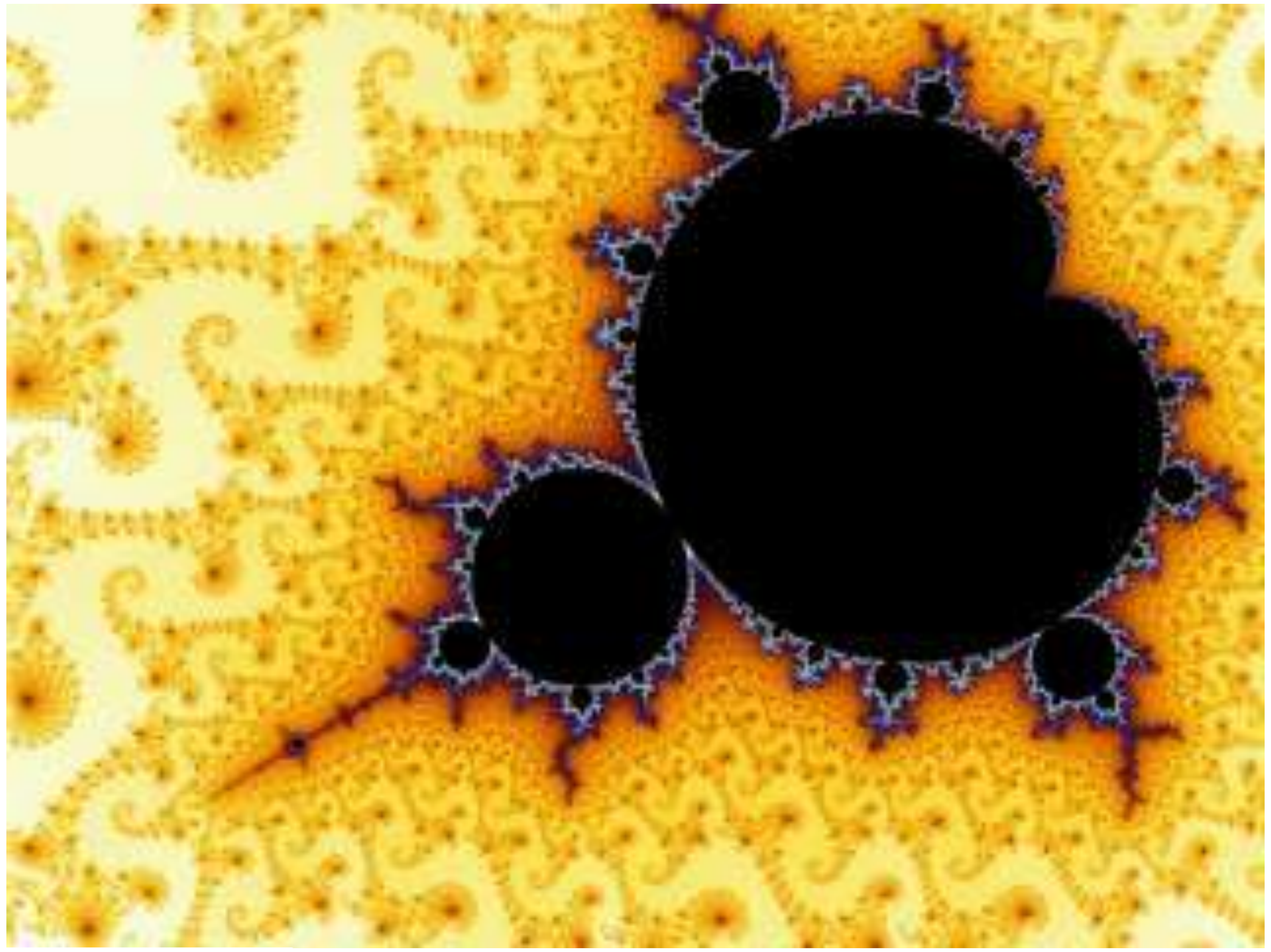
Labyrinth represented in Chartres Cathedral

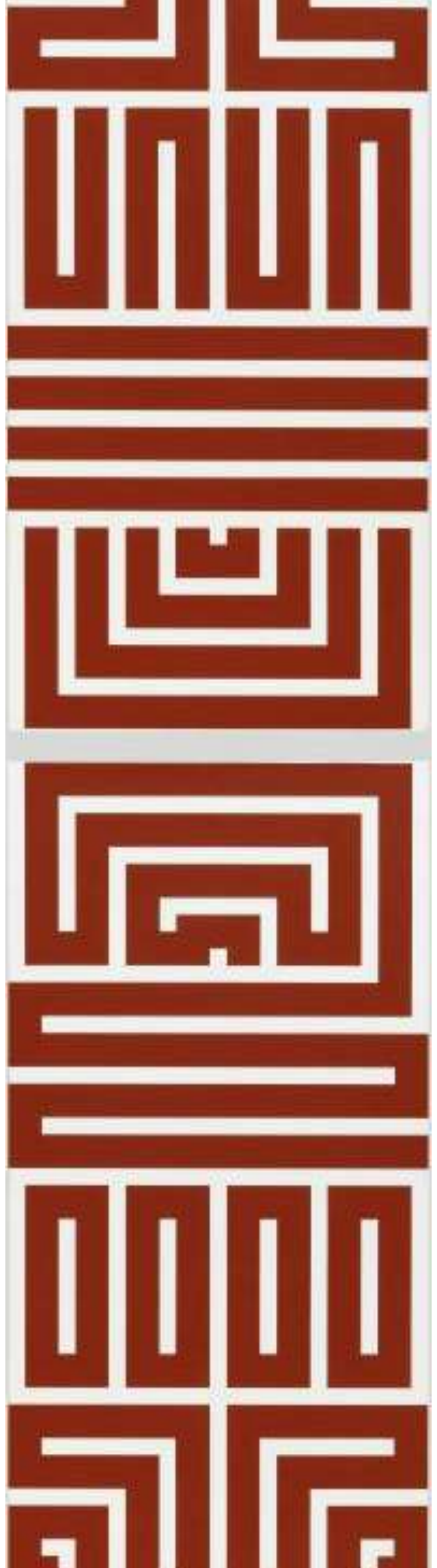
Face of war, Salvador Dali

MANDELBROT





$$\begin{cases} x_0 = x_{\text{pixel}} \qquad y_0 = y_{\text{pixel}} \\ x_{n+1} = x_n^2 - y_n^2 + c_x \\ y_{n+1} = 2x_n y_n + c_y \end{cases}$$



Vera Molnár

1924- 2023

02 TILING (TESSELATIONS)

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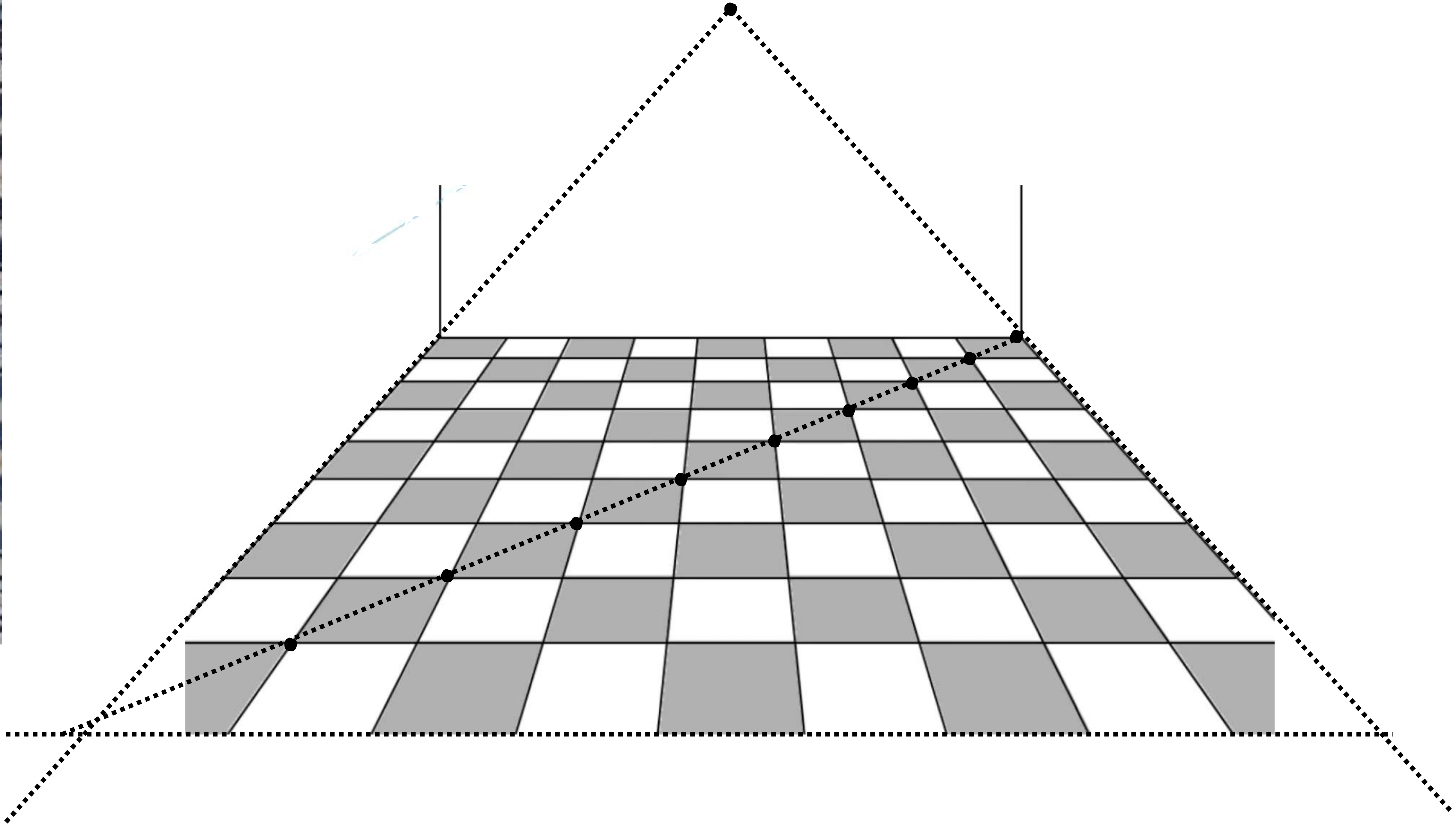
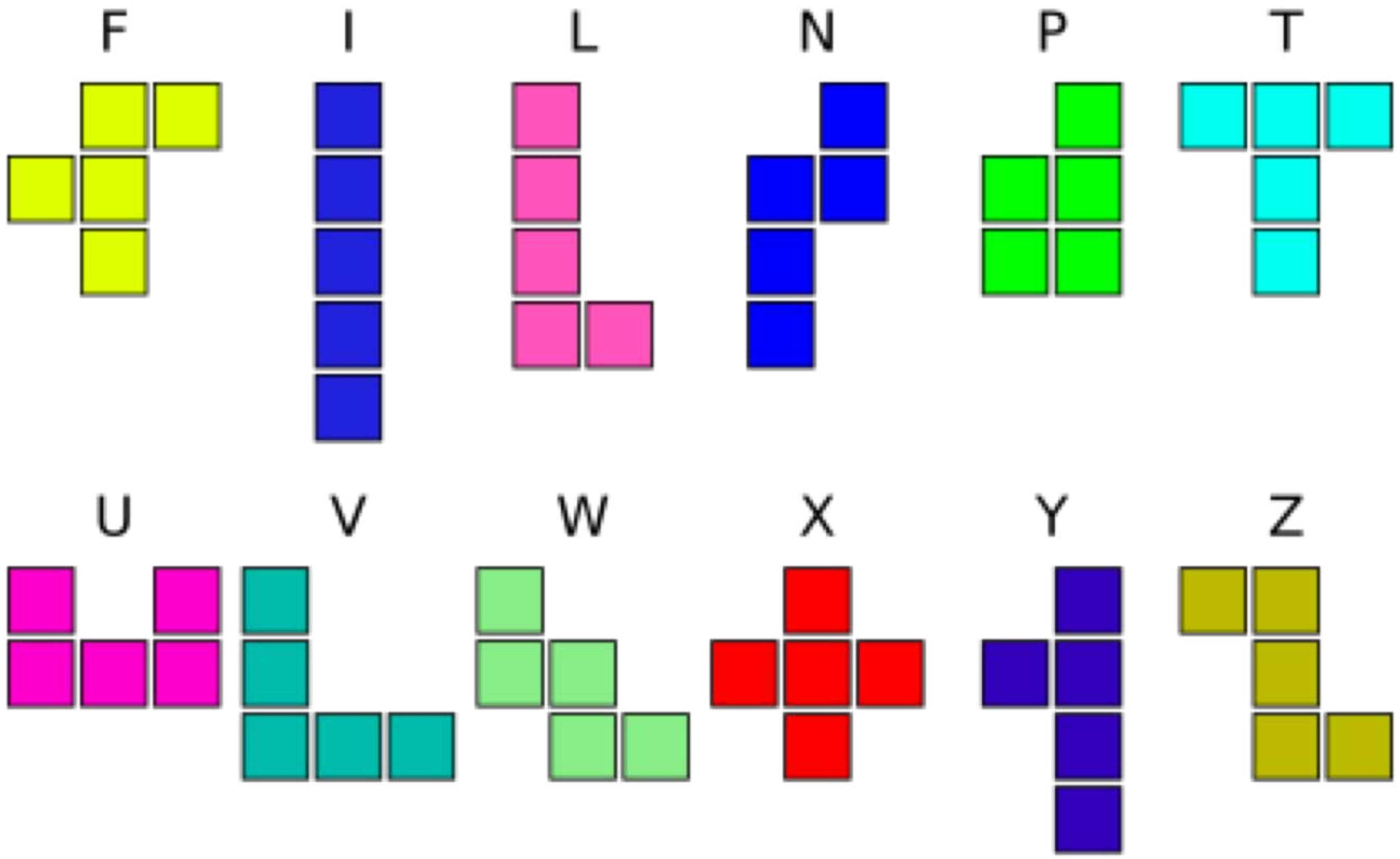
Covering of a surface, using one or more geometric shapes



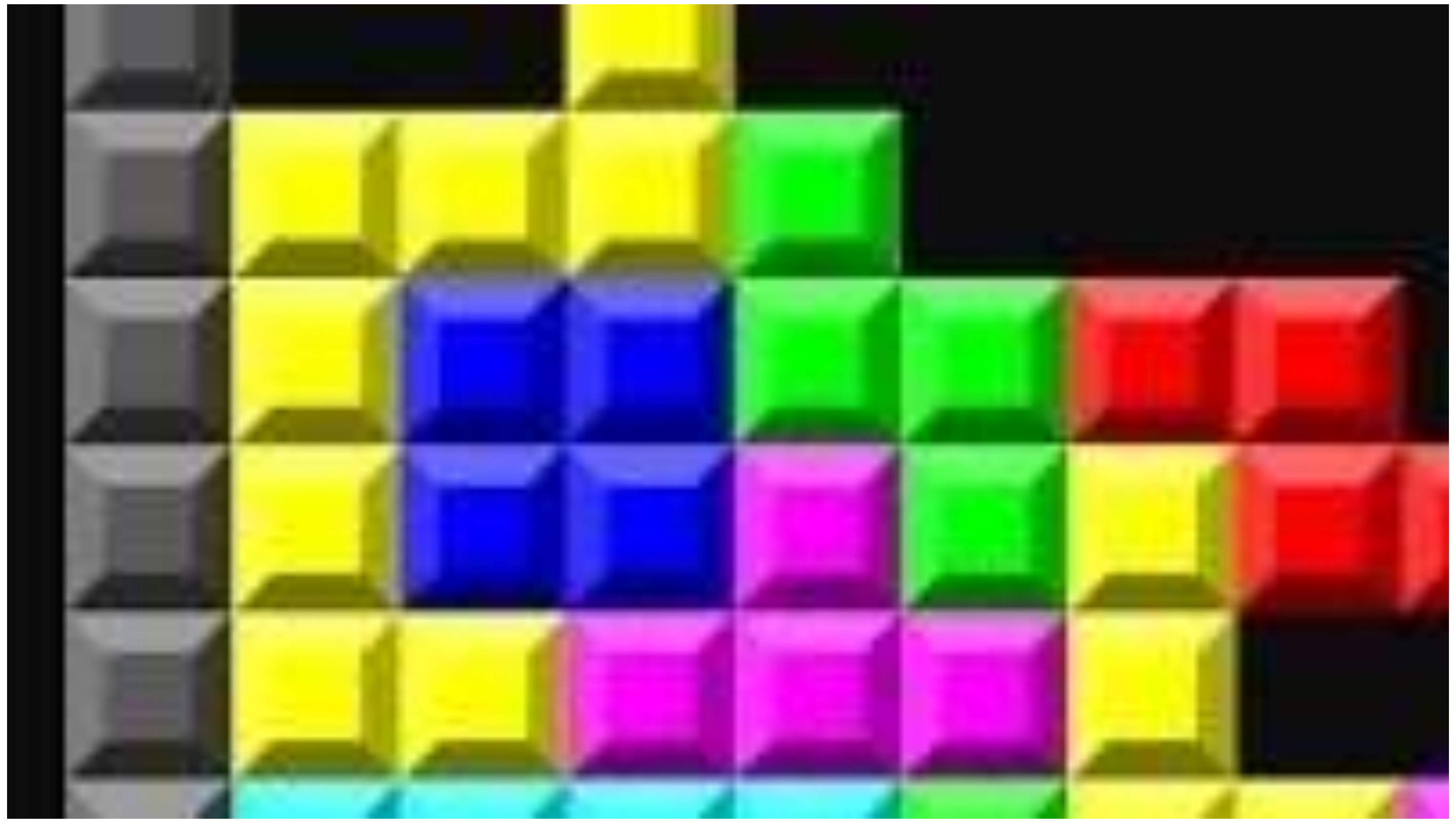
Pavement

Batik

OF WAX-RESIST DYEING APPLIED TO THE WHOLE CLOTH
(INDONESIA)



PENTOMINOES



What exactly is a tiling ???

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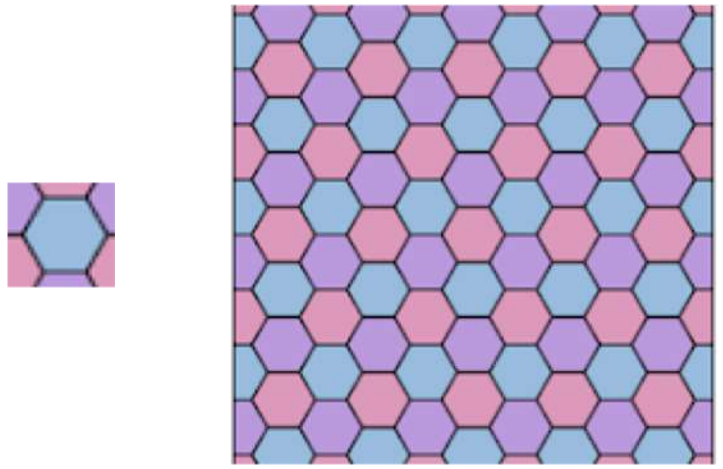
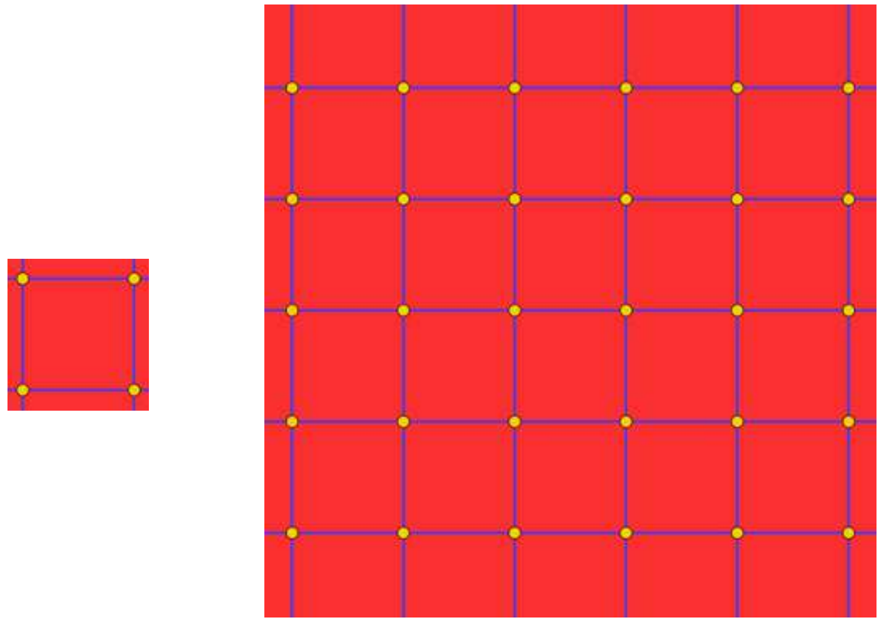
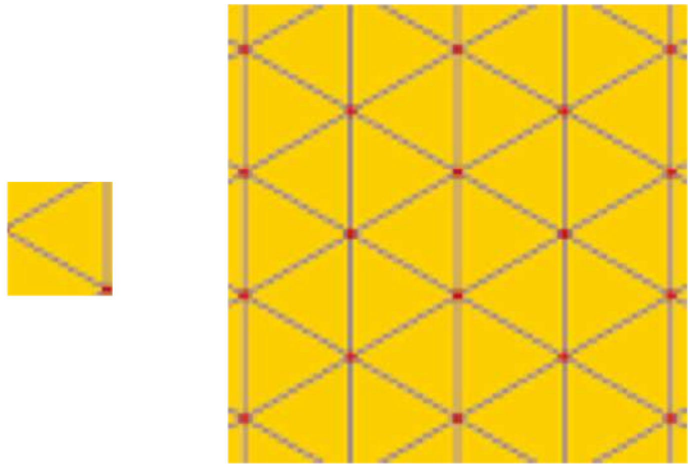
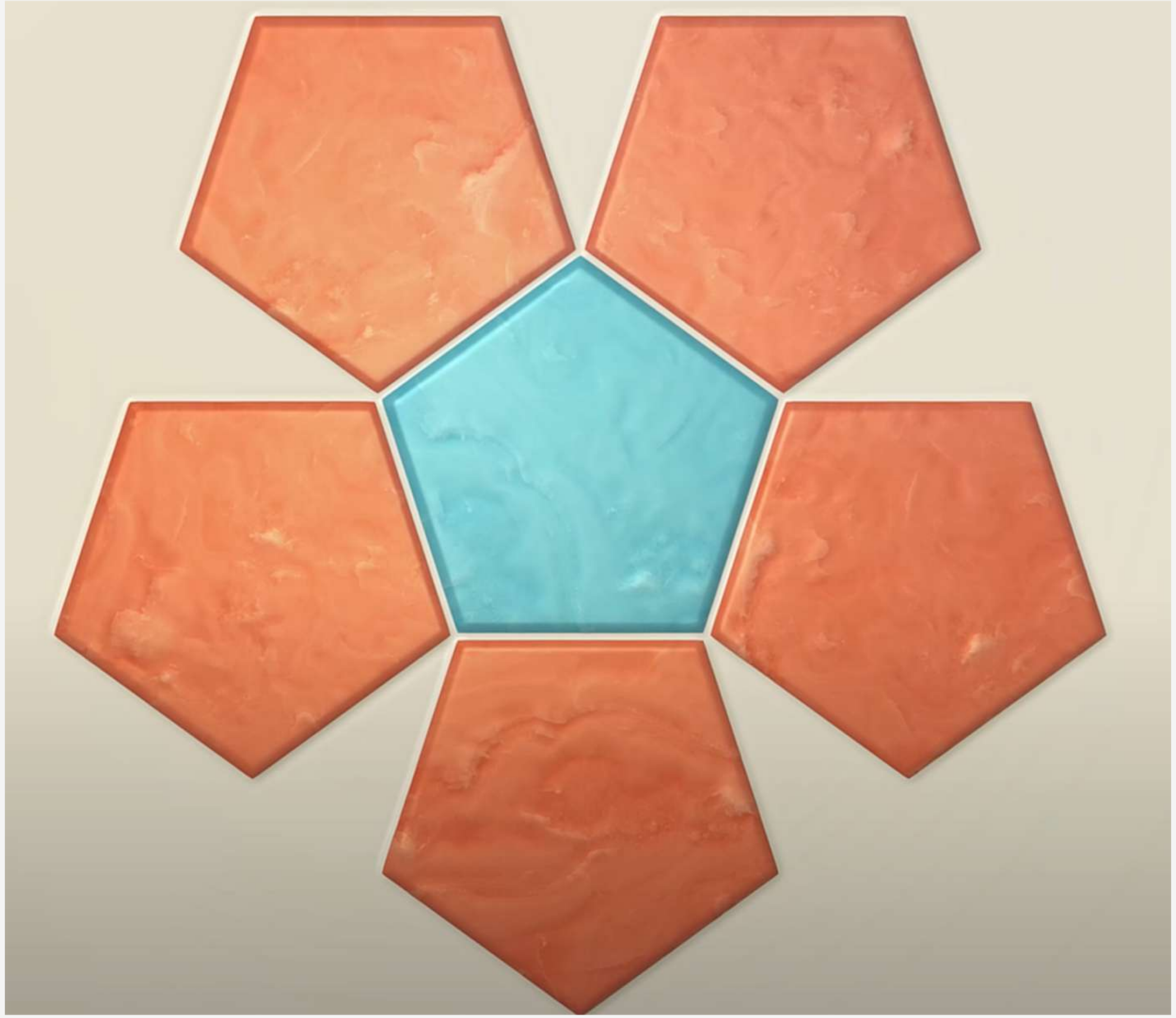
Tesselation or tiling = **Covering of a surface**, often a plane, using one or more geometric shapes, called **tiles**, with **no overlaps** and **no gaps**. In mathematics, tessellation can be generalized to higher dimensions and a variety of geometries.







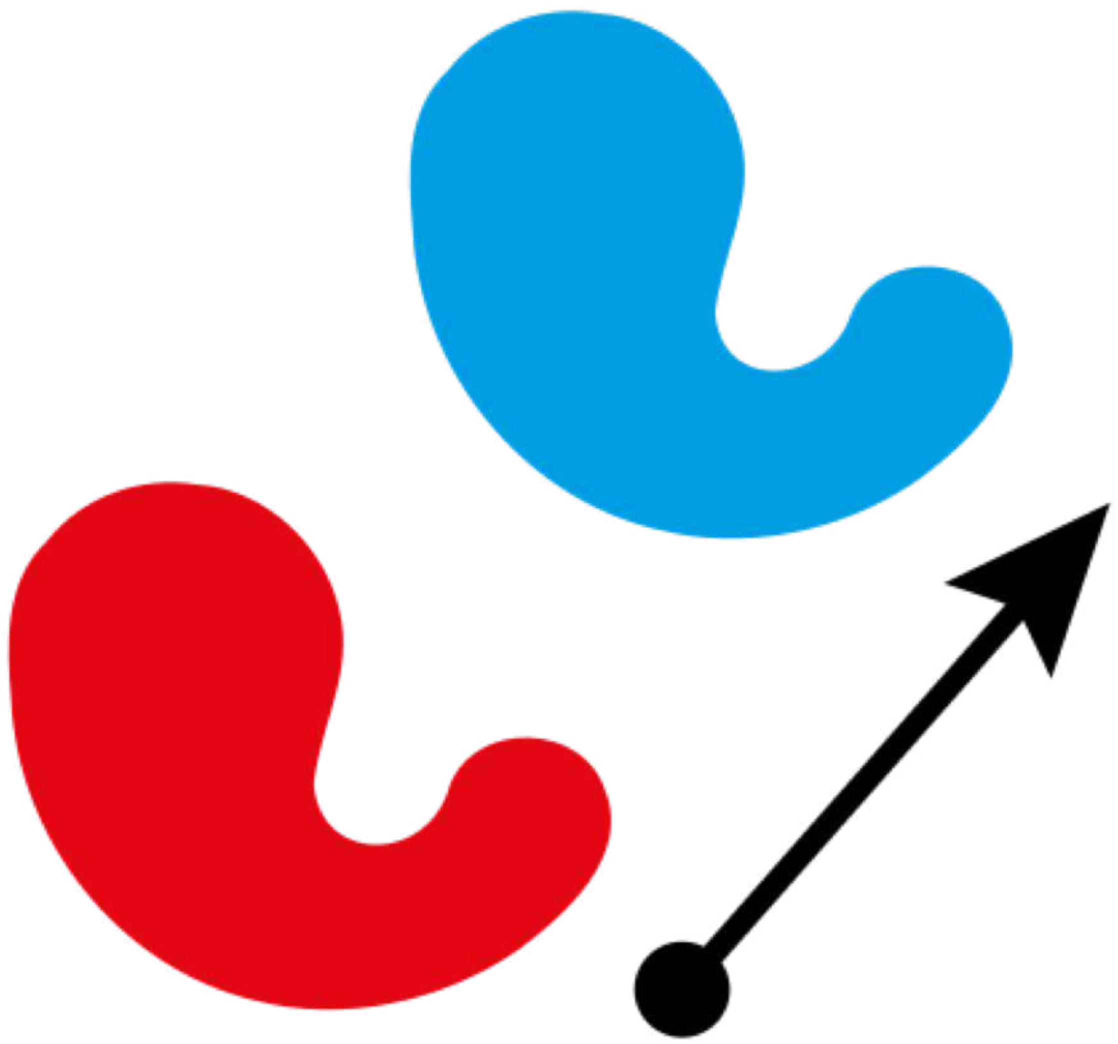
pentagons



Batik

OF WAX-RESIST DYEING APPLIED TO THE WHOLE CLOTH
(INDONESIA)

Translation= moves every point of a figure or a shape by the **same distance in a given direction**



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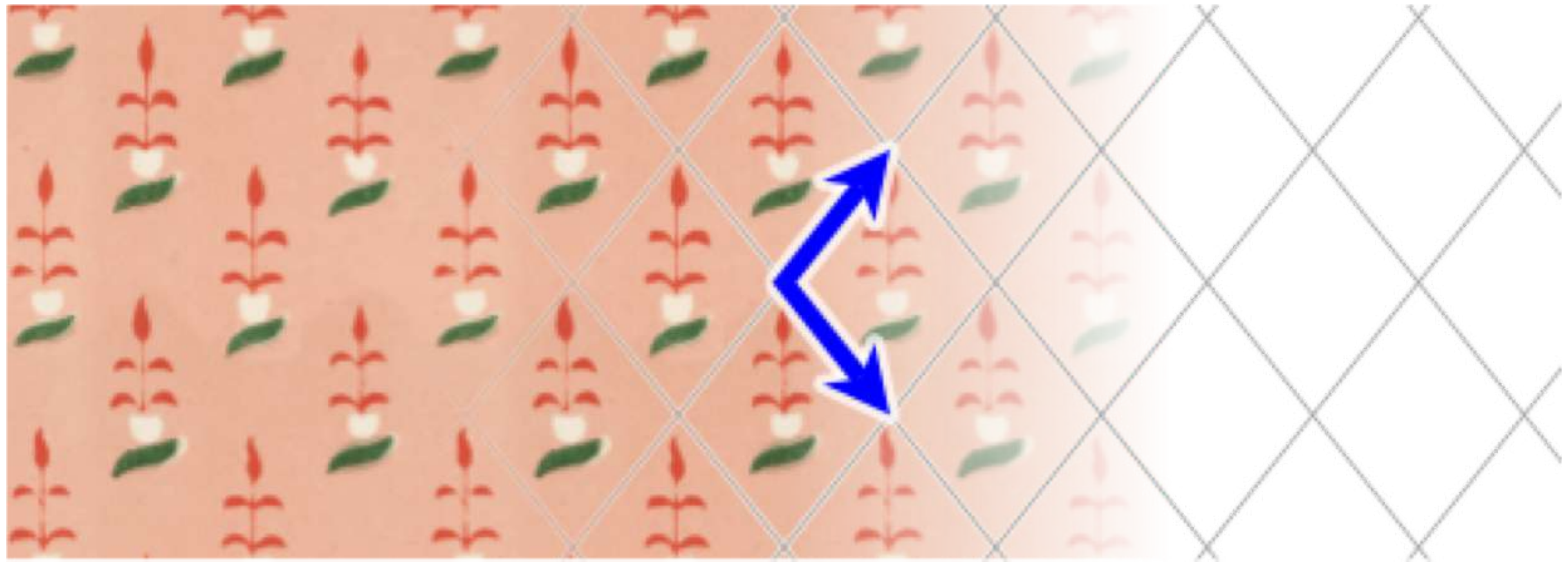

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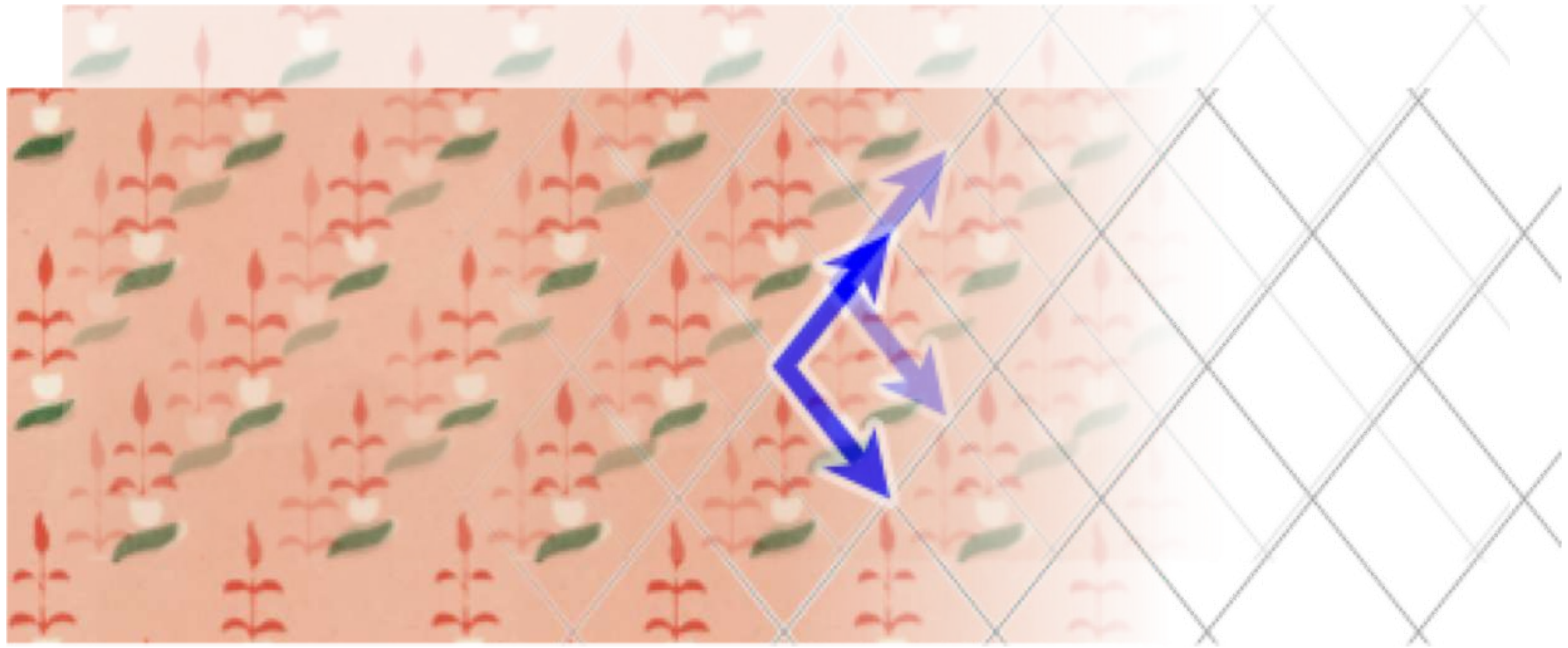

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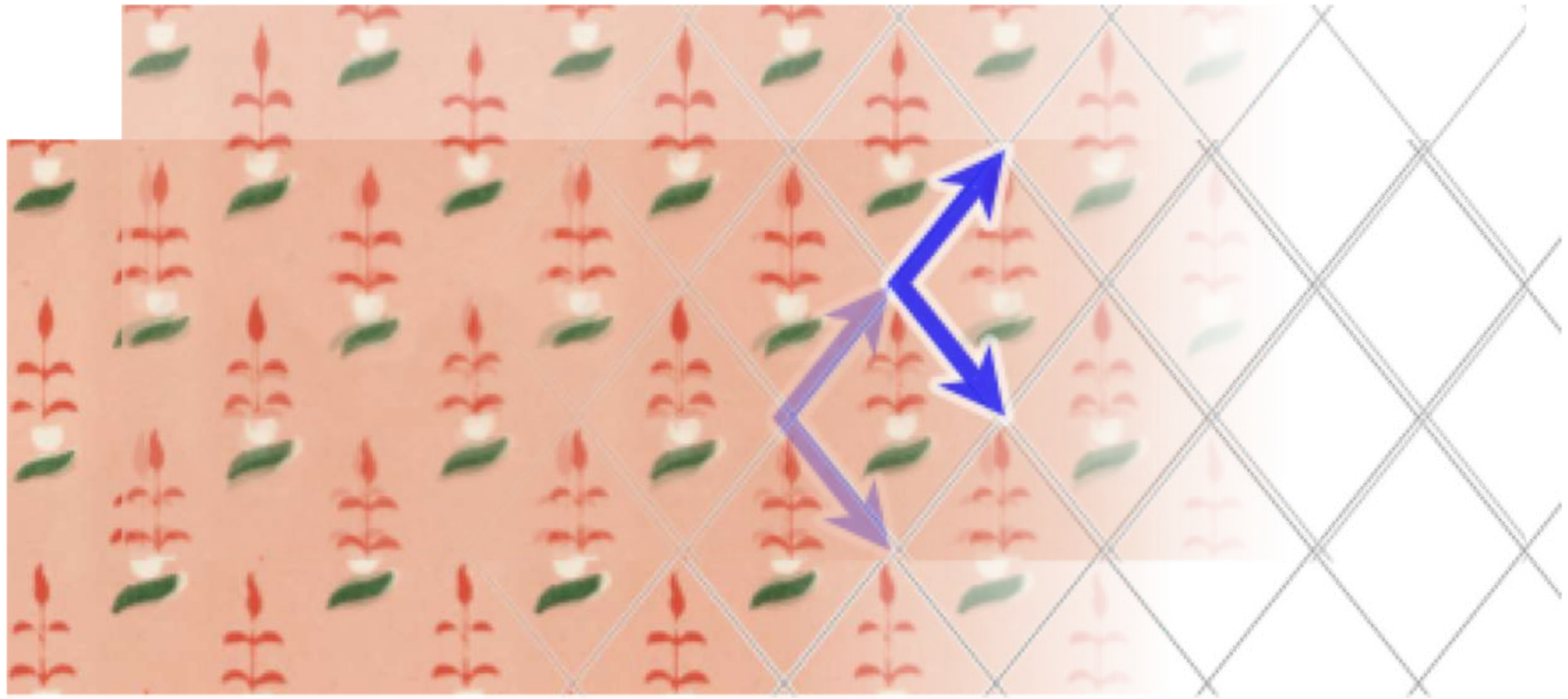
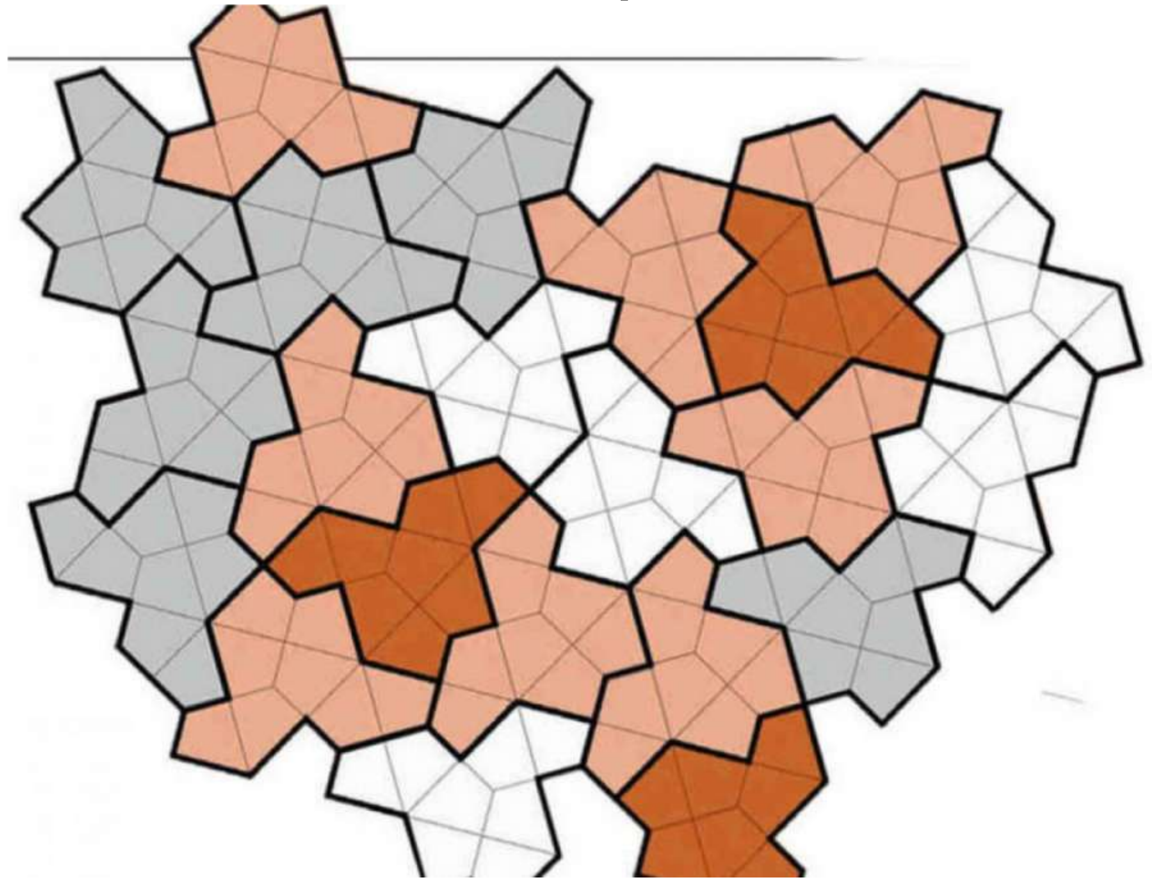


TESSELATIONS
Link with mathematics

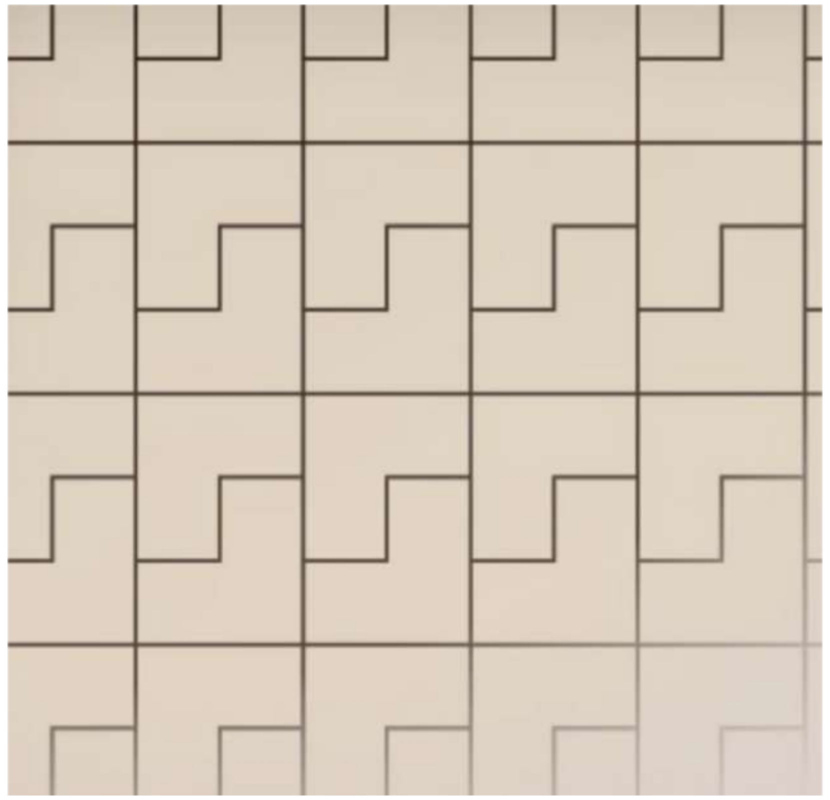
1. Periodic tiling: invariance by translation
2. Aperiodic tiling

Aperiodic



Einstein
problem
(2023)

Periodic



- **Periodic** = There must be **2 translations independant** (Bieberbach theorem)

- **Aperiodic** = Otherwise

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Isometries of the Euclidean plane

- Translations
- Rotations
- Reflections
- Glide reflections

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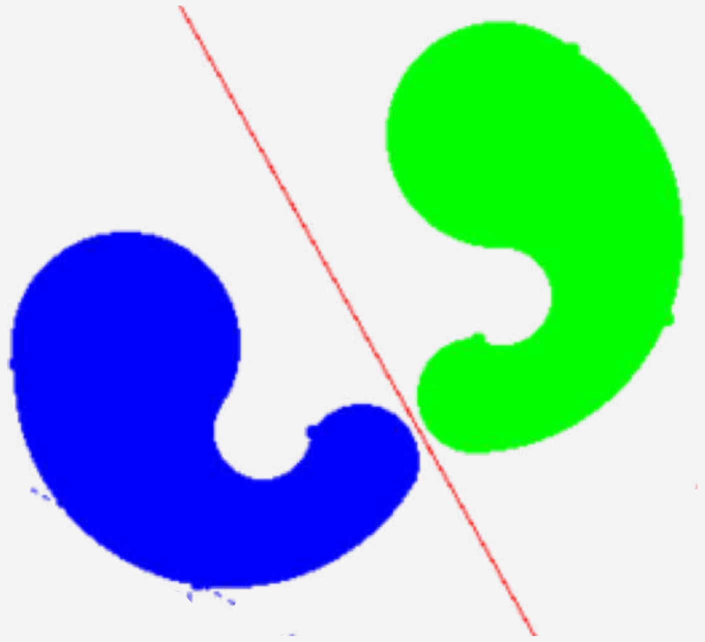

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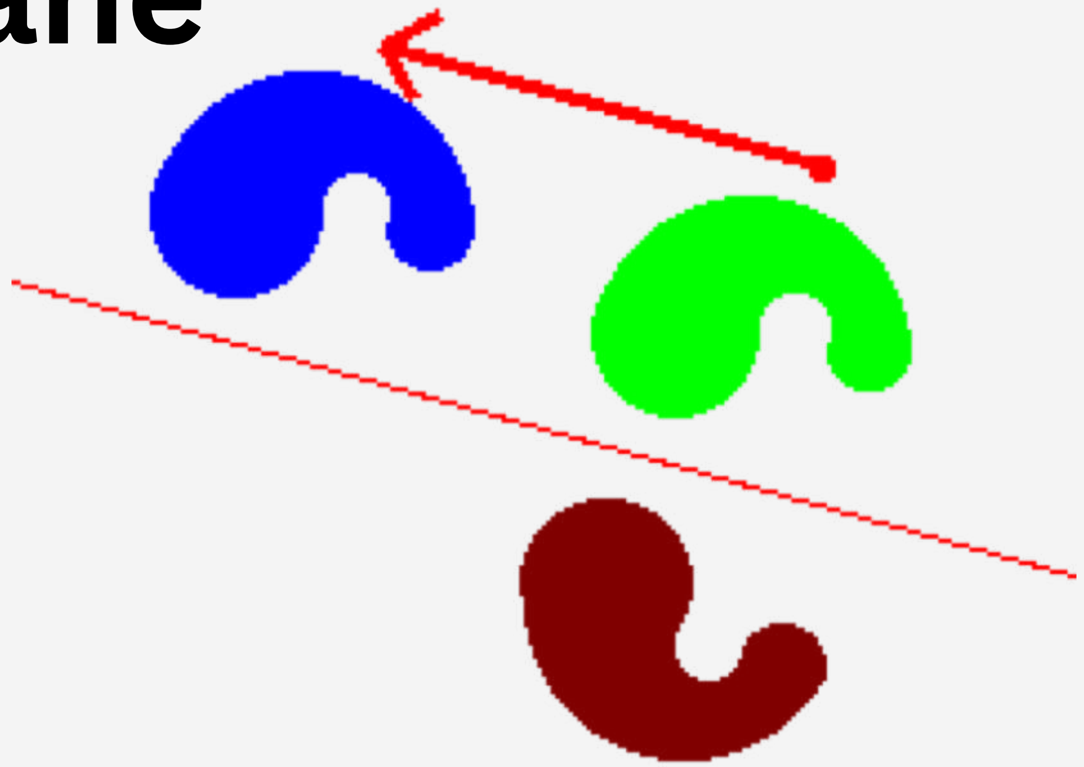
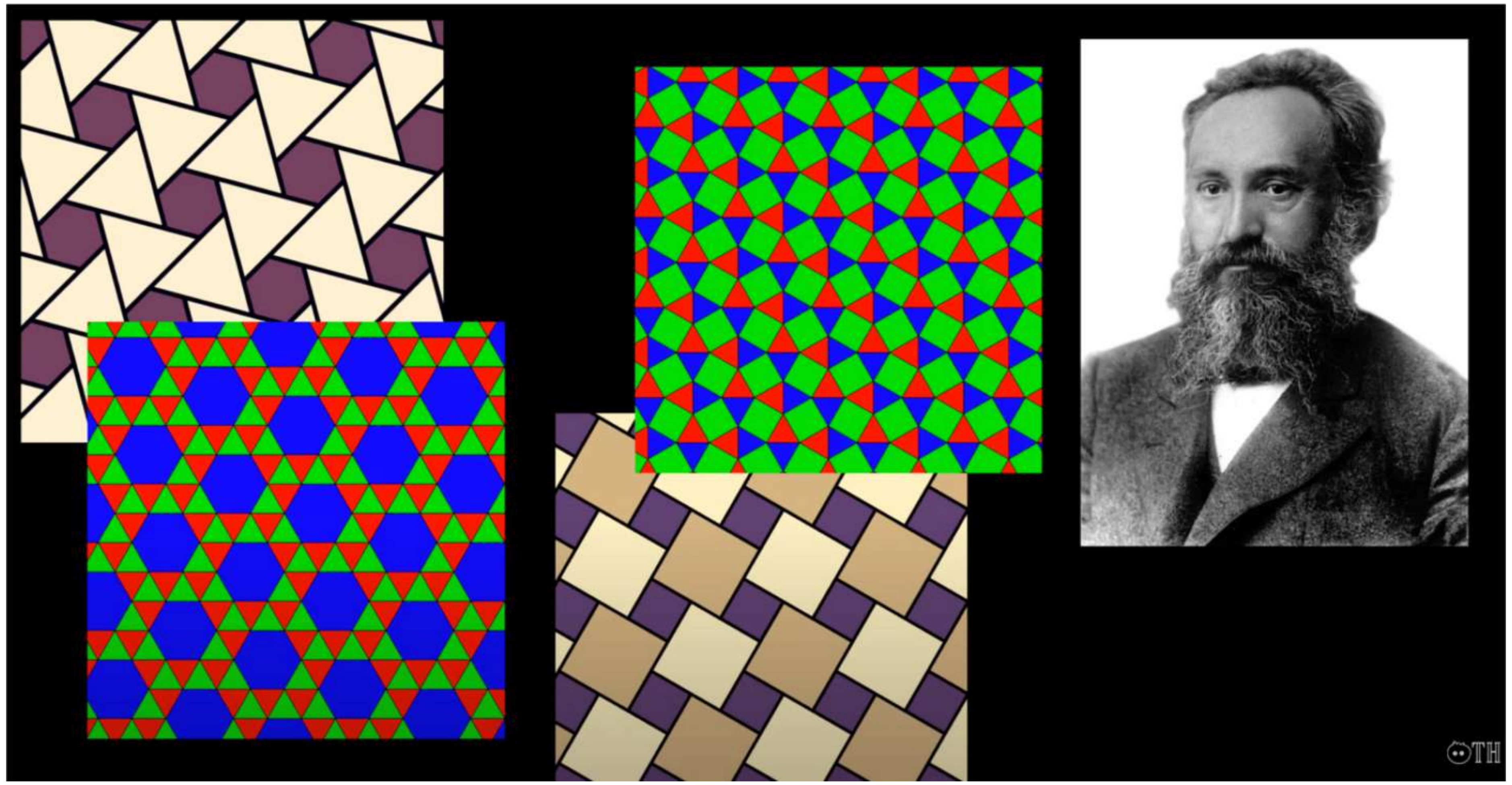


TILING
Link with mathematics

1. **Periodic tiling** : invariance par translation

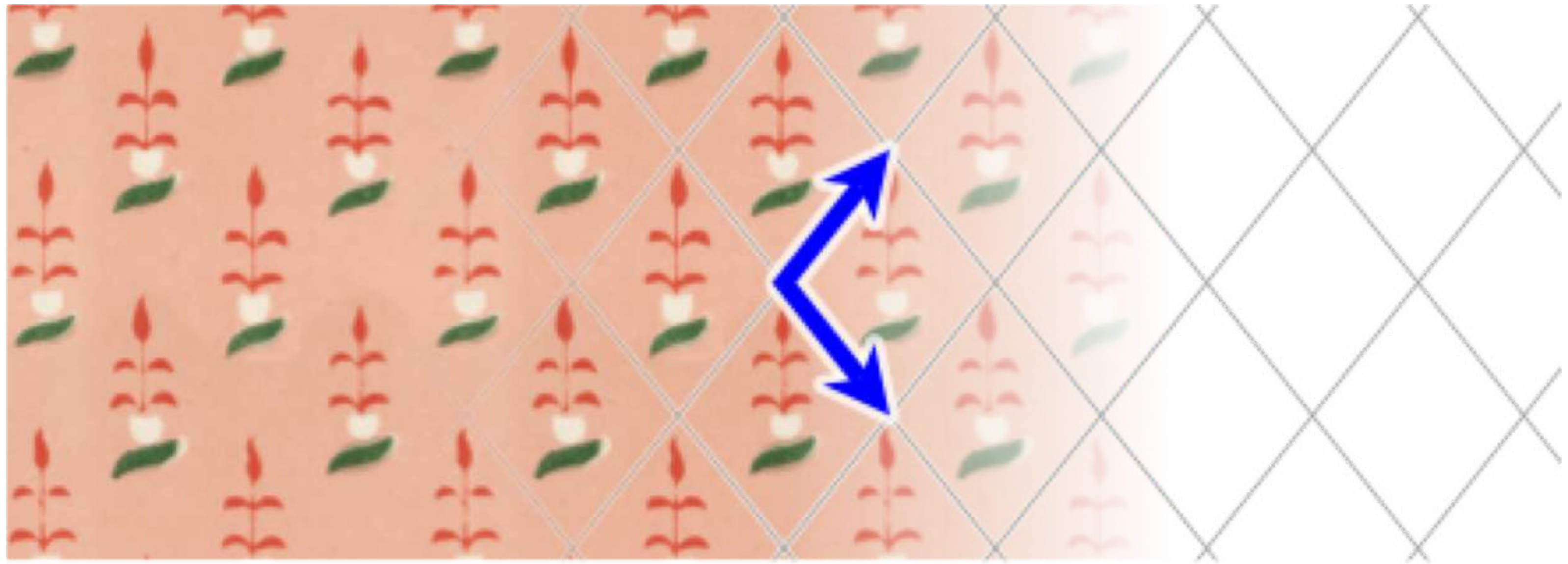
STEPANOVICH FEDOROV SHOWS THAT WITH THIS GEOMETRY, THERE EXISTS 17 KINDS OF PERIODIC TILINGS (1891)

Size of smallest rotation	Has reflection?			
	Yes		No	
360° / 6	*p6m* (*632)		*p6* (632)	
360° / 4	Has mirrors at 45°?		*p4* (442)	
	Yes: *p4m* (*442)	No: *p4g* (4*2)		
360° / 3	Has rot. centre off mirrors?		*p3* (333)	
	Yes: *p31m* (3*3)	No: *p3m1* (*333)		
360° / 2	Has perpendicular reflections?		Has glide reflection?	
	Yes	No		
	Has rot. centre off mirrors?	*pmg* (22*)	Yes: *pgg* (22×)	No: *p2* (2222)
	Yes: *cmm* (2*22) / No: *pmm* (*2222)			
none	Has glide axis off mirrors?		Has glide reflection?	
	Yes: *cm* (*×)	No: *pm* (**)	Yes: *pg* (××)	No: *p1* (o)



- Translations
- Rotations
- Reflections
- Glide reflections

P1 GROUP



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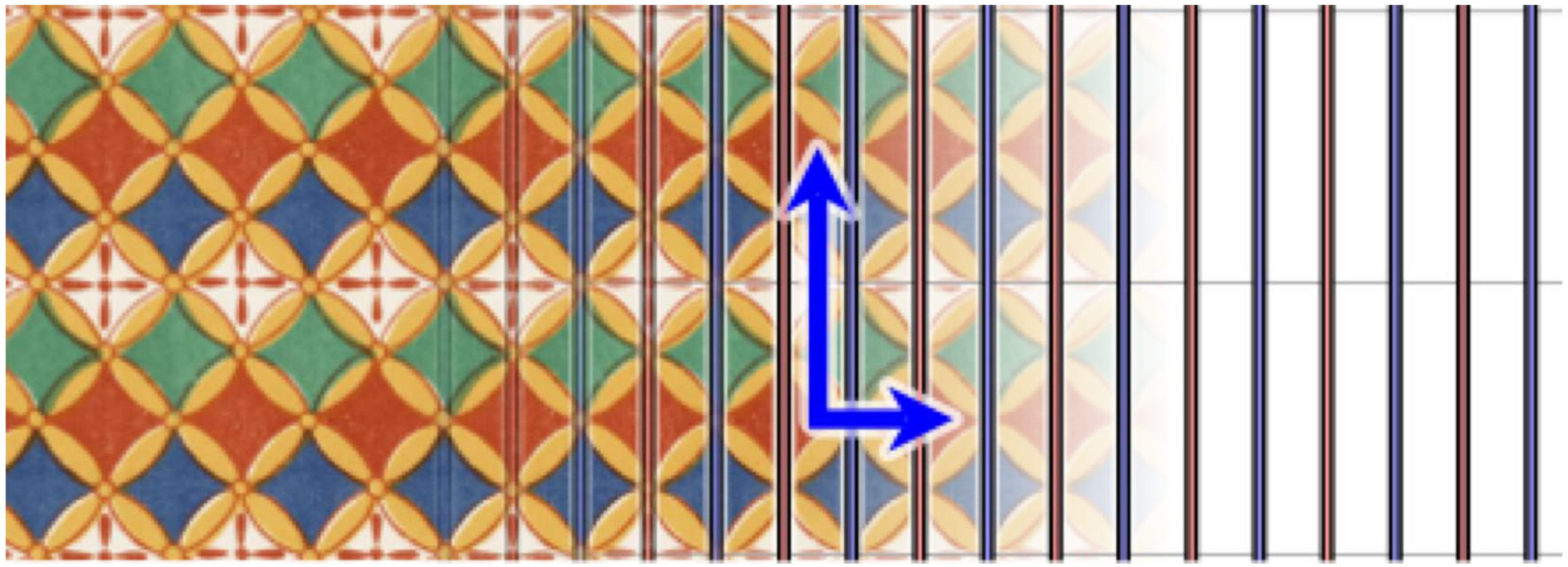
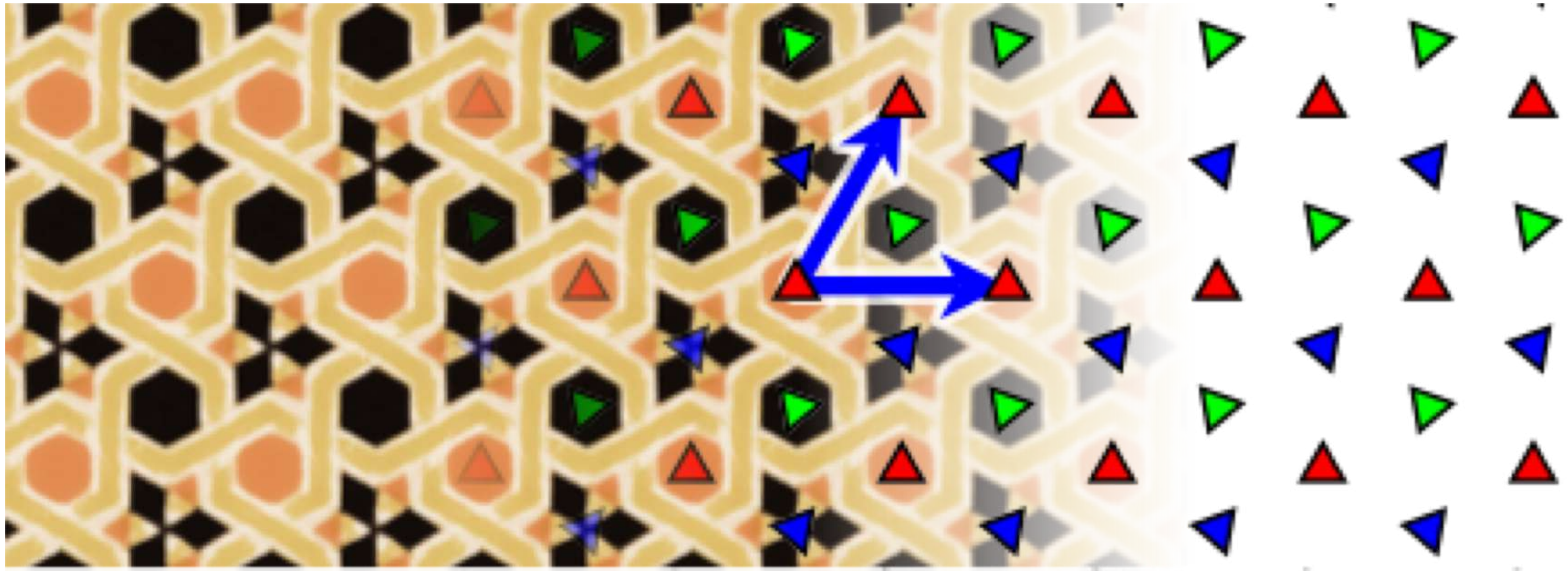
- **Translations**
- Rotations
- Reflections
- Glide reflections

PM GROUP



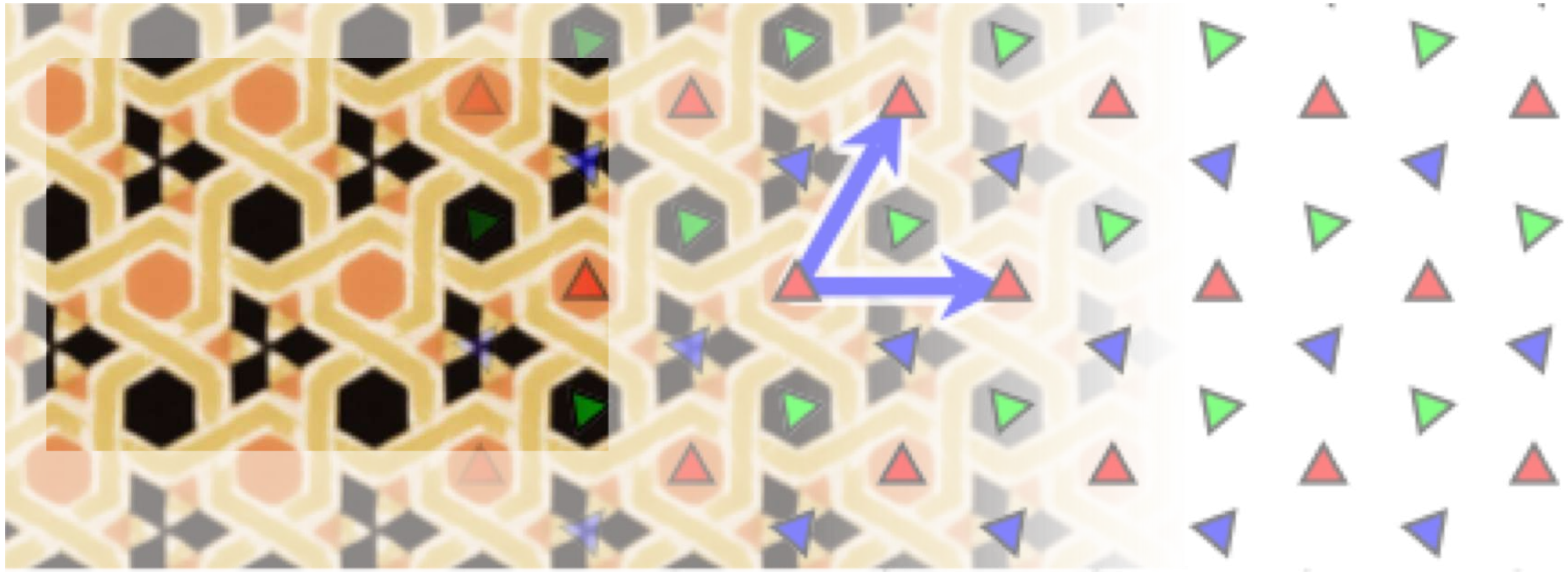
- **Translations**
- Rotations
- **Reflections**
- Glide reflections

P3 GROUP



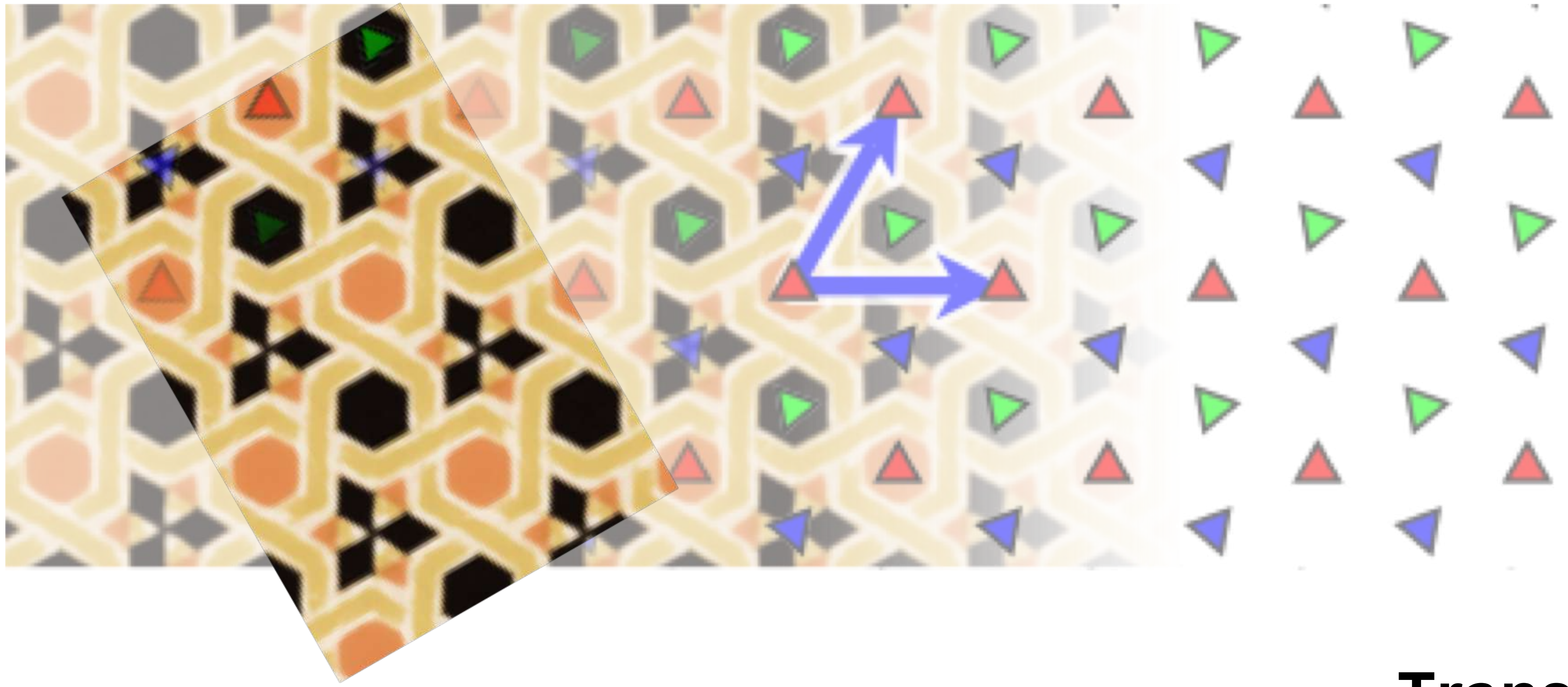
- **Translations**
- **Rotations**
- Reflections
- Glide reflections

P3 GROUP



- **Translations**
- **Rotations**
- Reflections
- Glide reflections

P3 GROUP

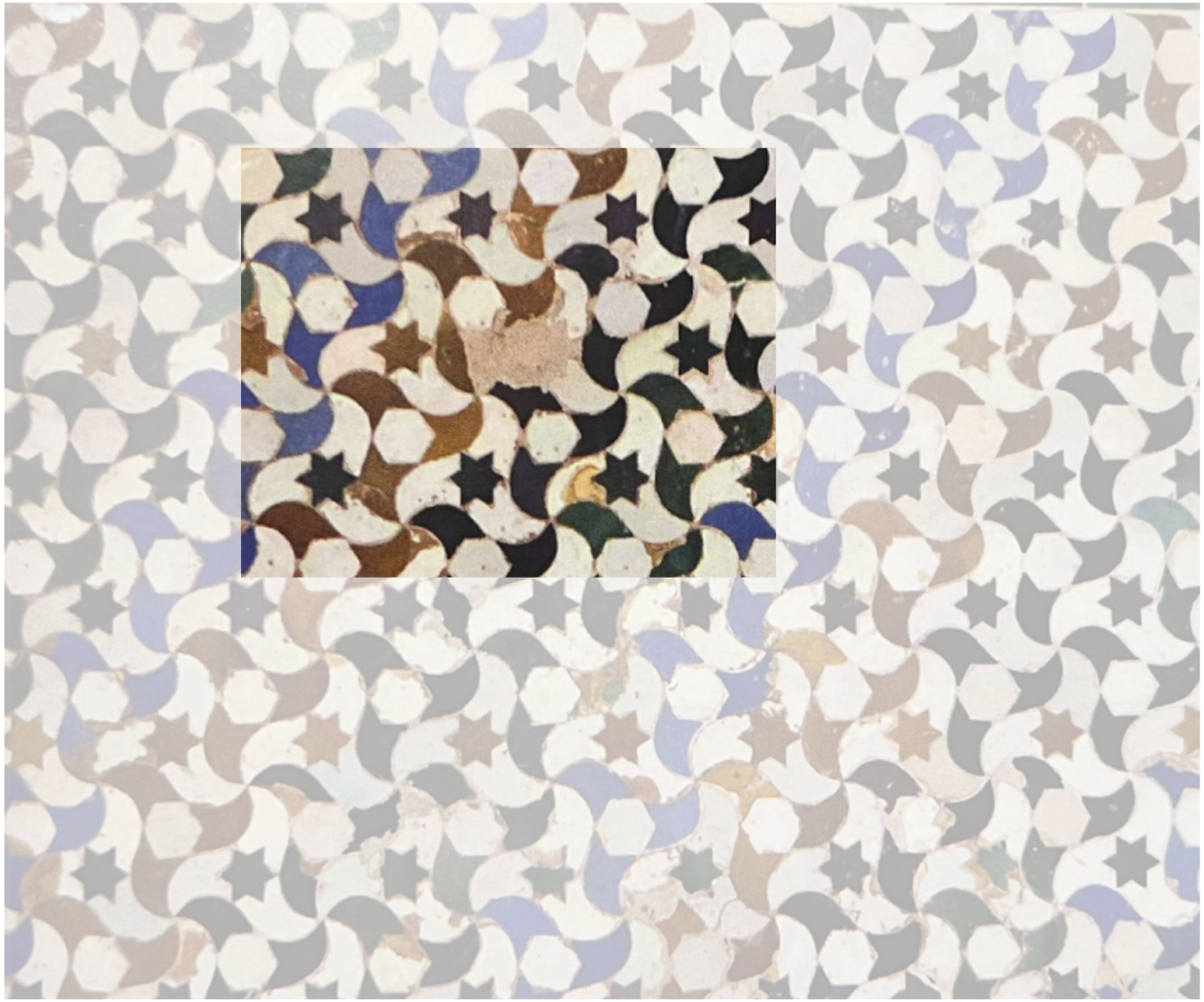


- **Translations**
- **Rotations**
- Reflections
- Glide reflections

P3 GROUP

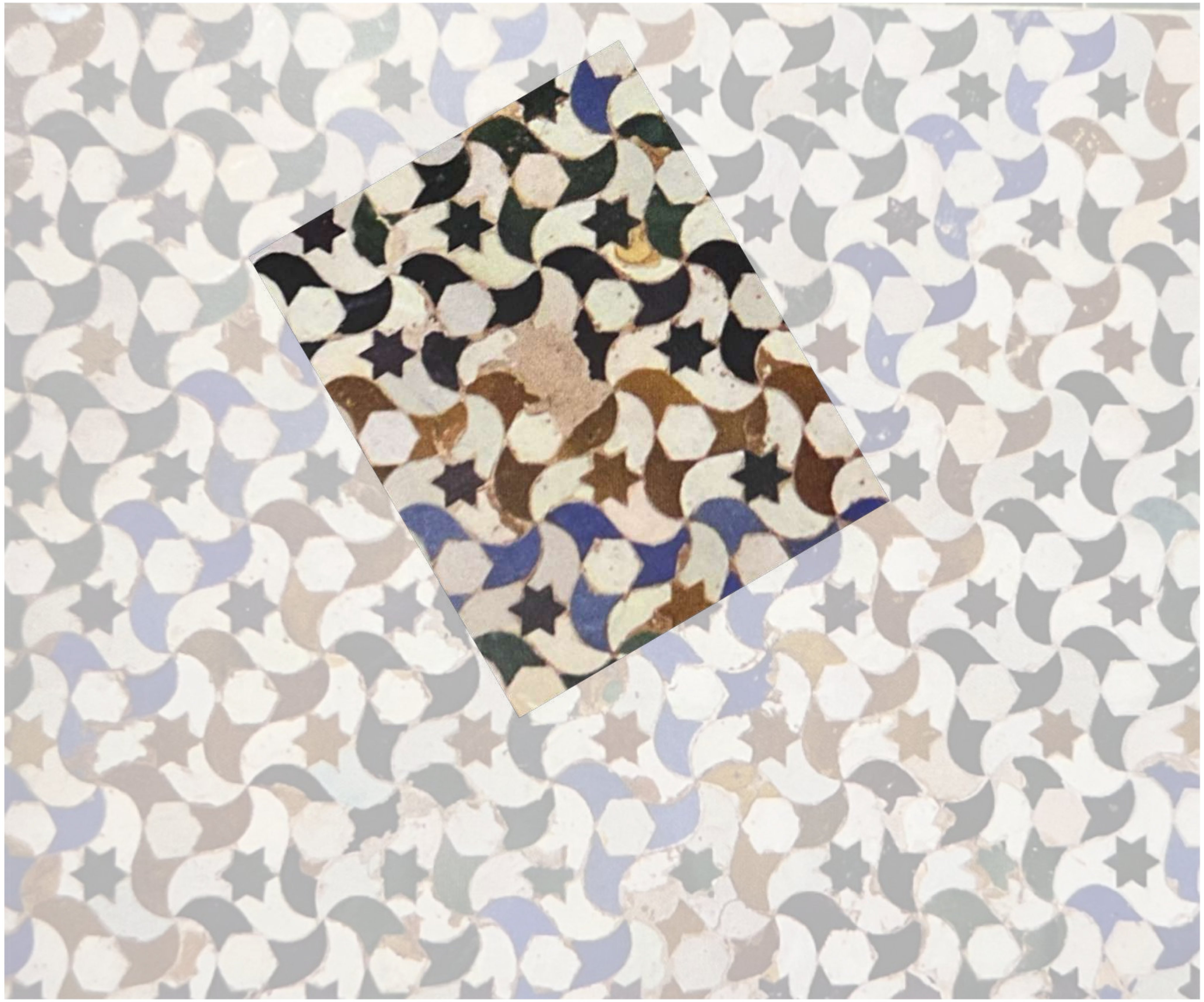
- Translations
- Rotations
- Reflexions
- Glide reflexions

P3 GROUP



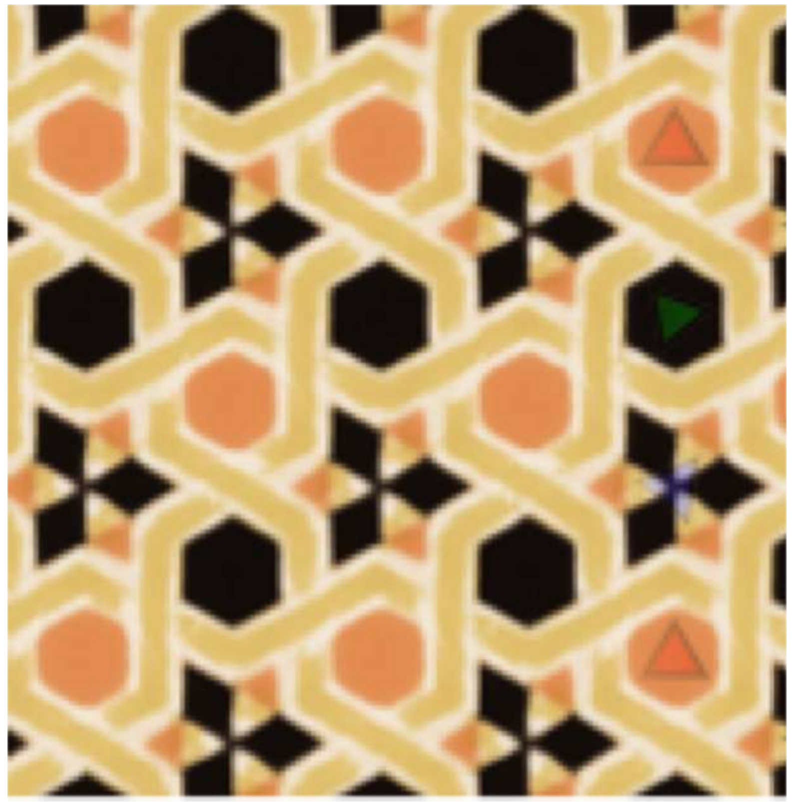
- **Translations**
- **Rotations**
- Reflections
- Glide reflections

P3 GROUP



- **Translations**
- **Rotations**
- Reflections
- Glide reflections

P3 GROUP



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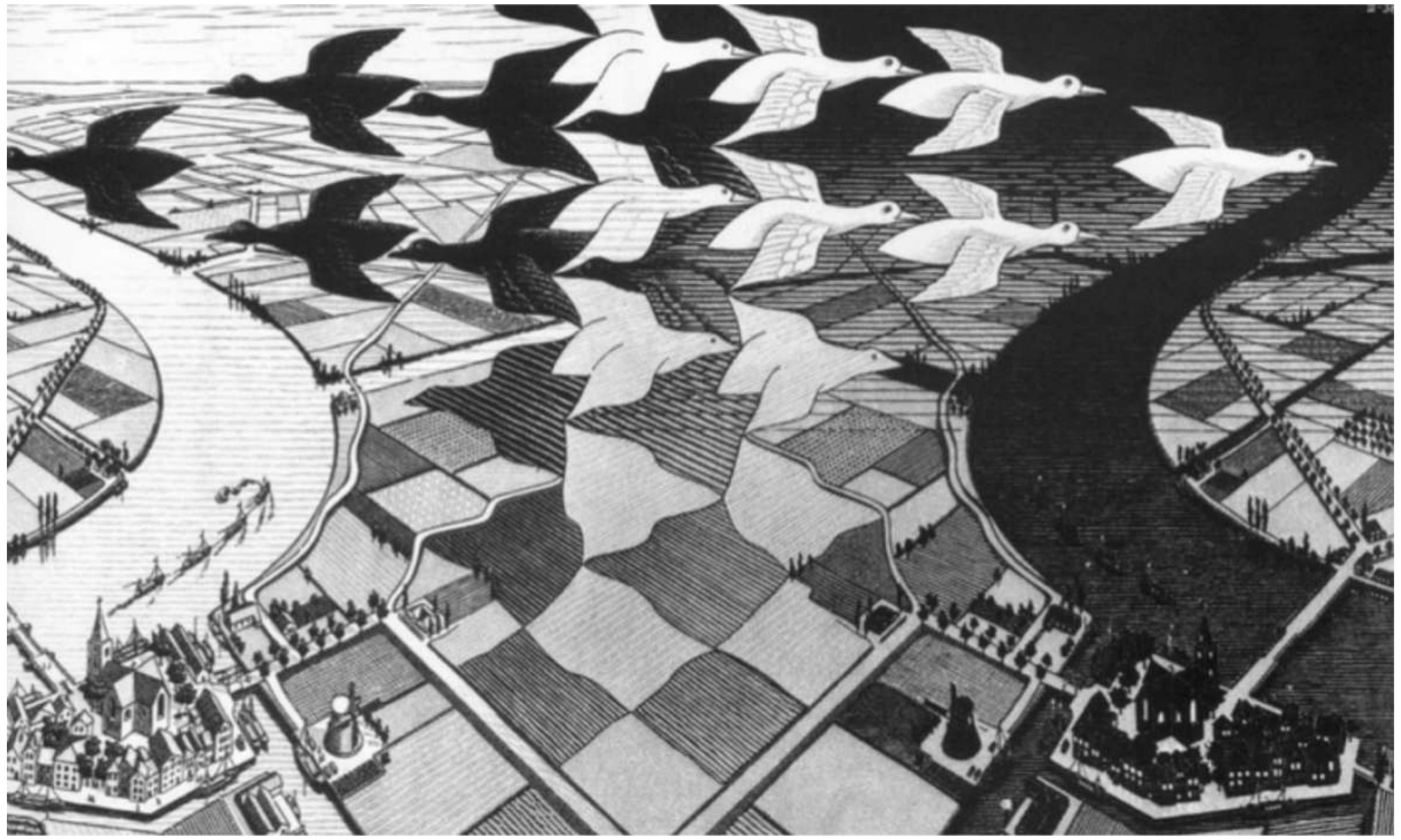
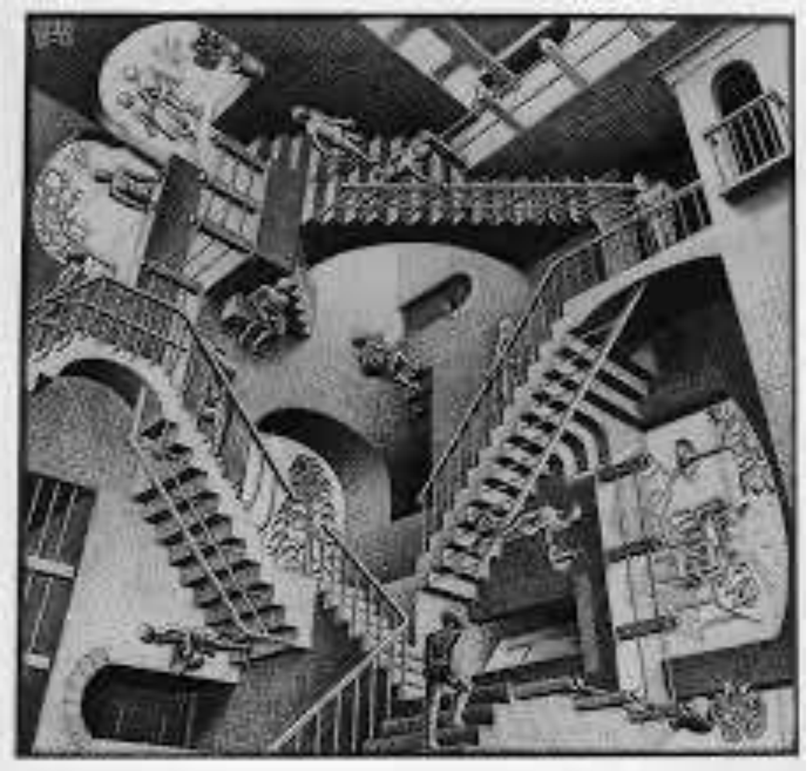
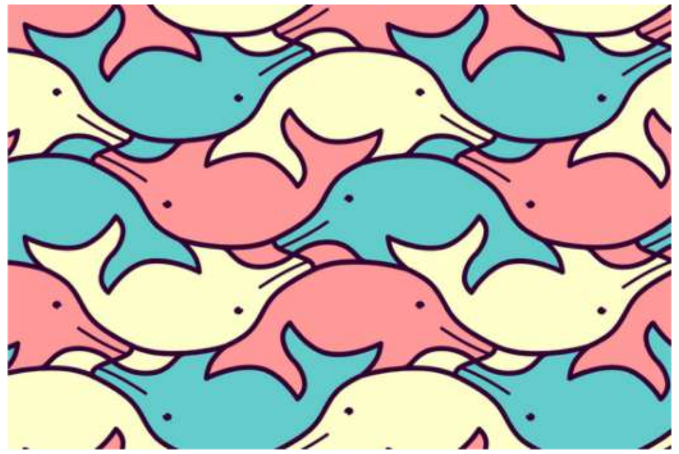
- **Translations**
- **Rotations**
- Reflections
- Glide reflections

Maurits Cornelis Escher
1898-1972





https://tiled.art/en/home/



TILING

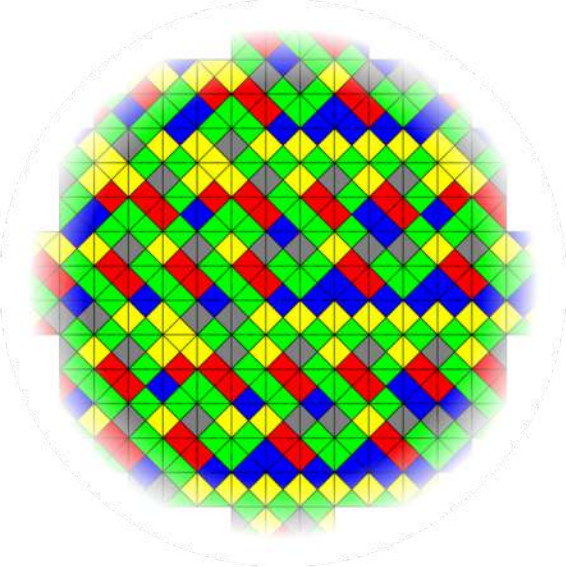
Aperiodic tiling

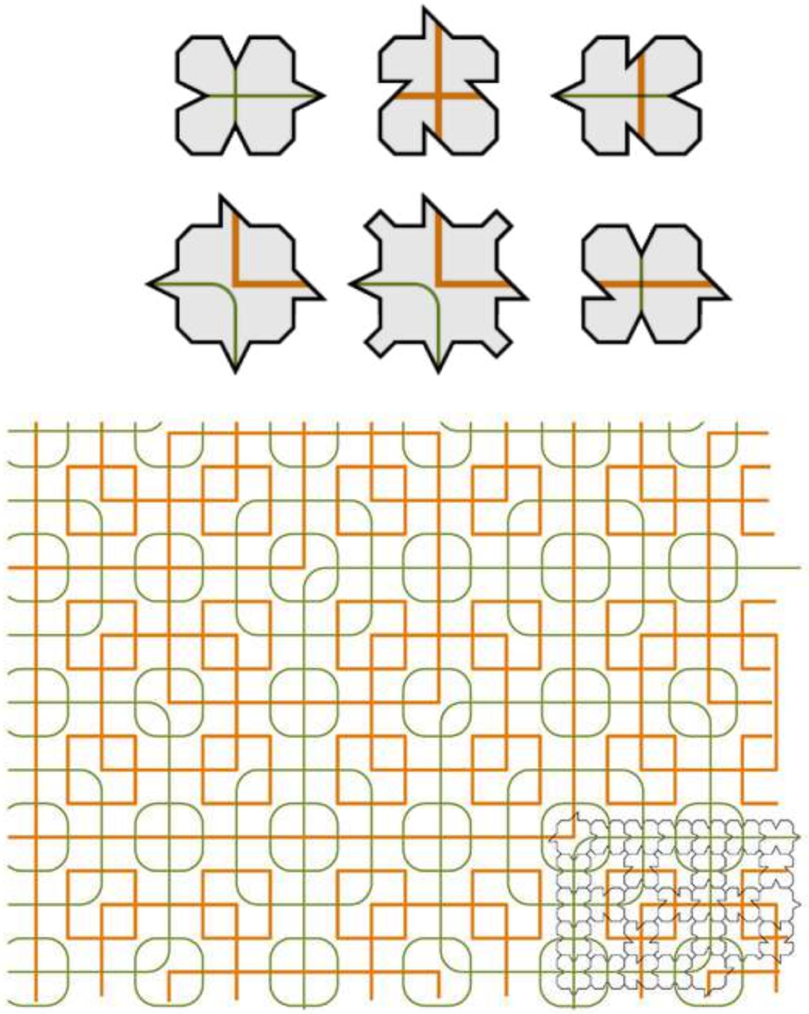
2. **Aperiodic tiling**

- In 1961, Wang tiles (dominoes) are first proposed by Hao Wang
- In 1966, Robert Berger found a set of 20 426 tiles that could only aperiodically tasselate the 2d plane
- In 1974, Roger Penrose found a set of 6 tiles
- In 1996, Karel Culik and Jarkko Kari found a set of 13 dominoes tiles

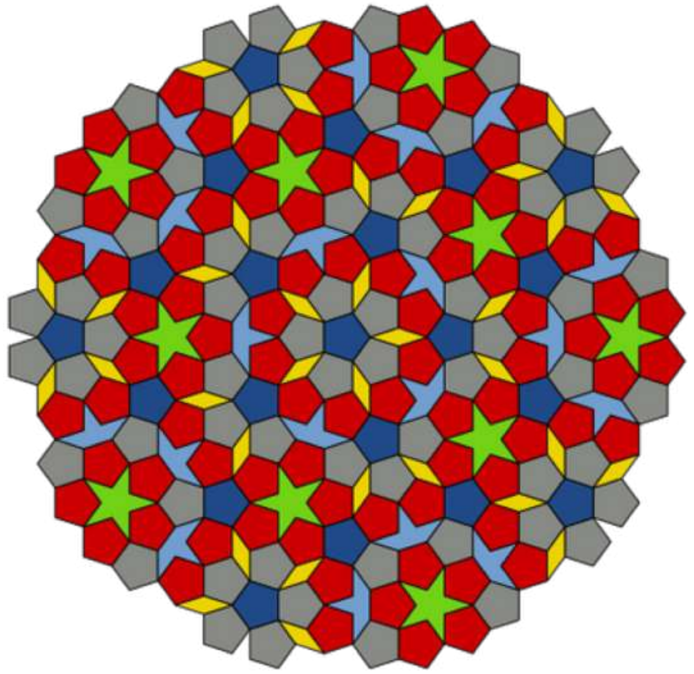
...

- In 2023, the first tiling non-periodic with a unique tile

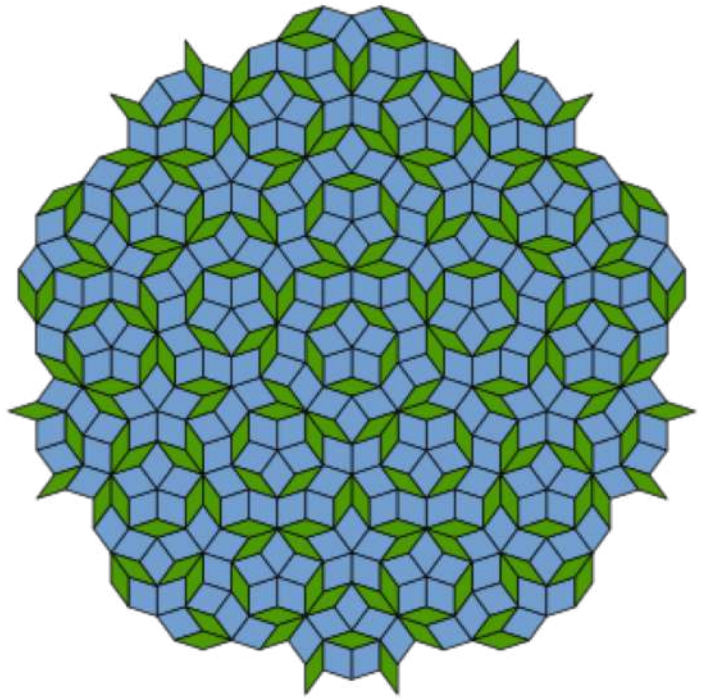




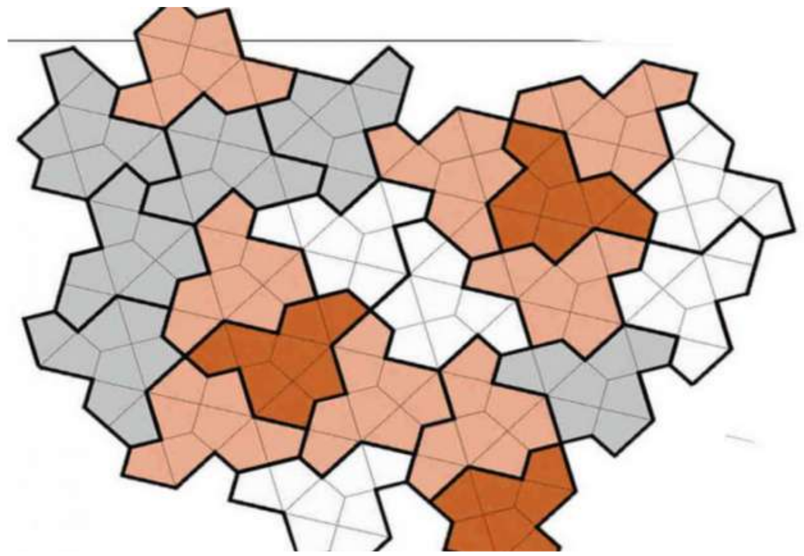
Robinson tiles



Penrose tiling

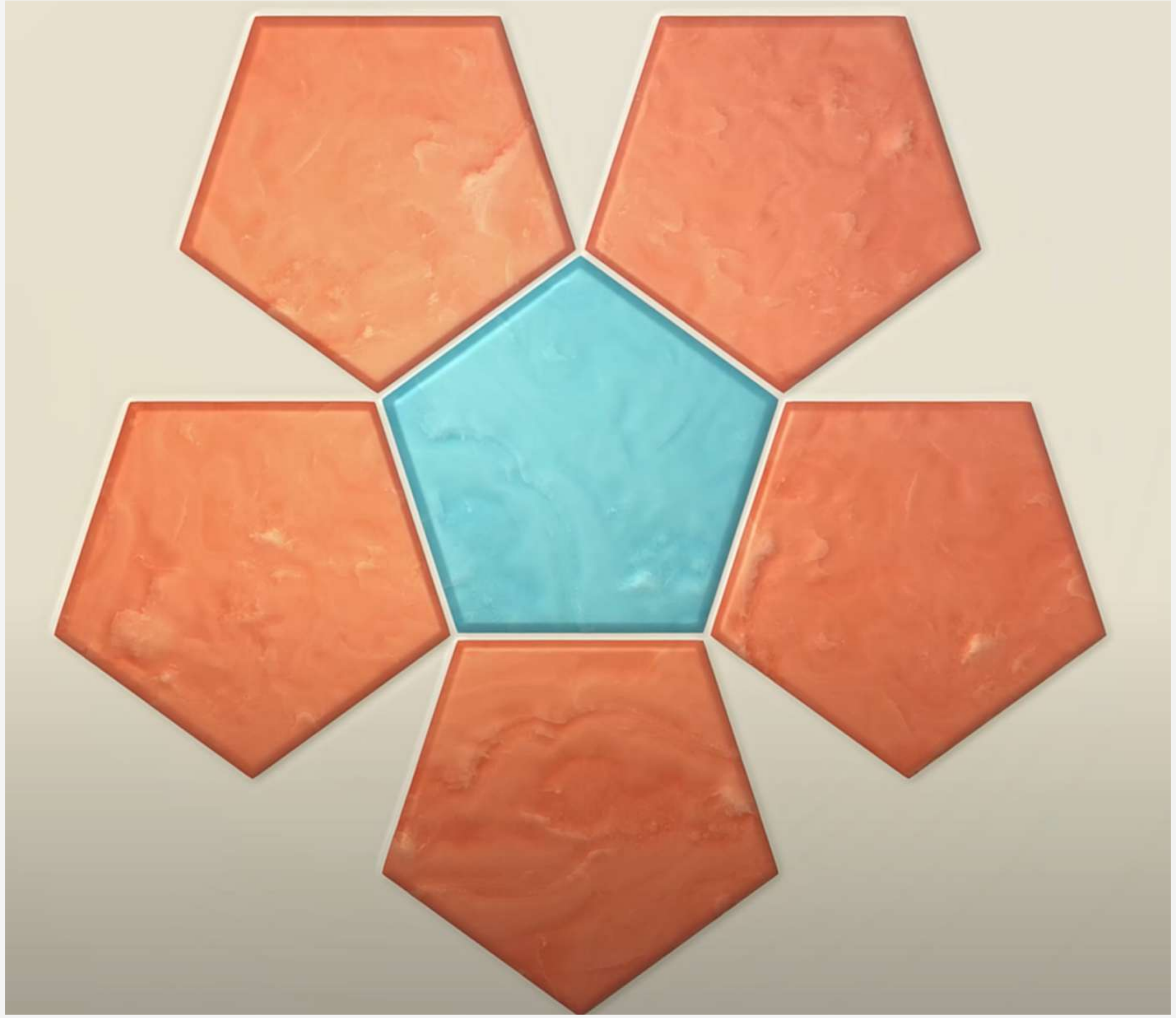


Penrose tiling

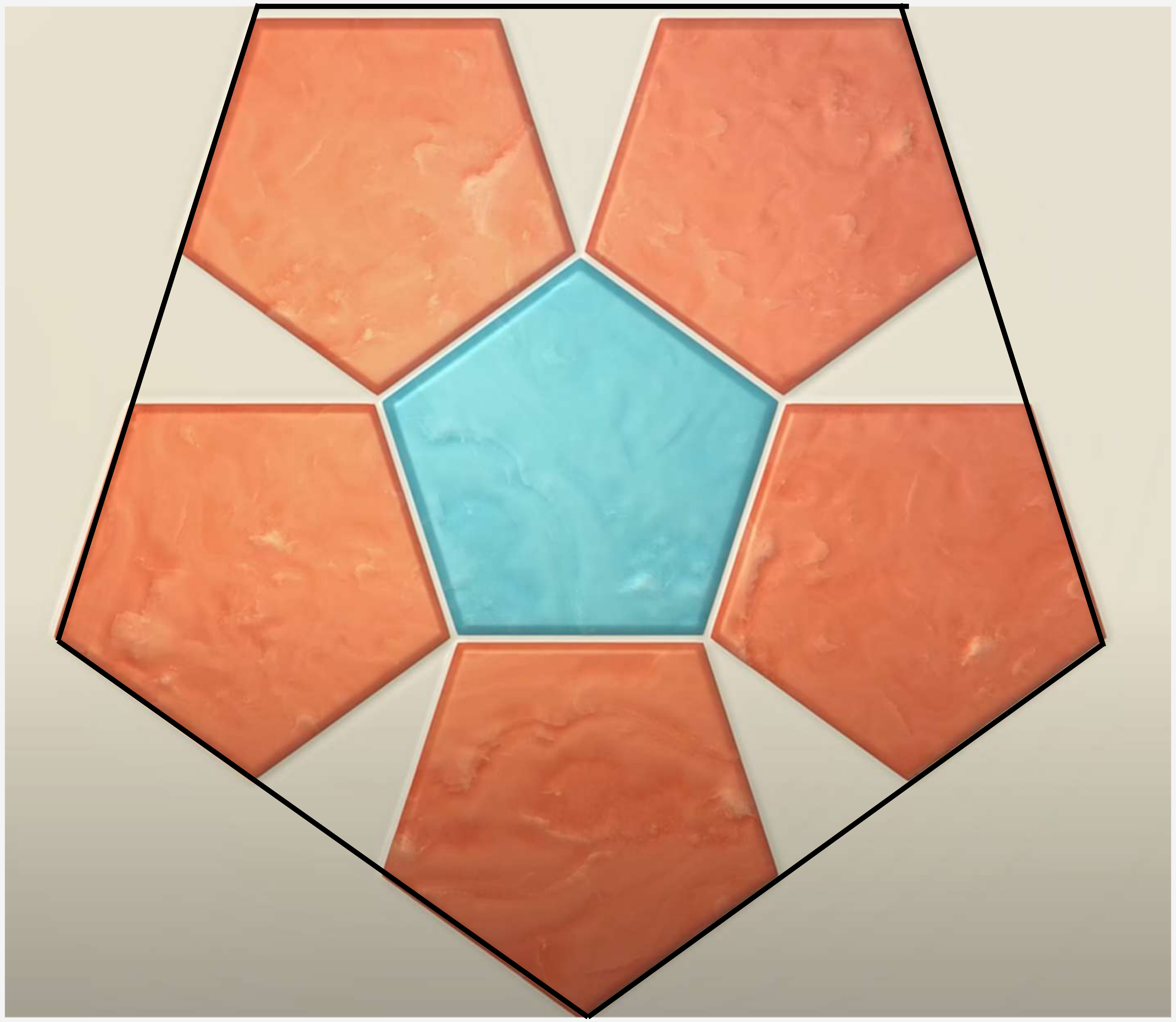


Einstein problem

These pentagons form a larger pentagon



These pentagons form a larger pentagon



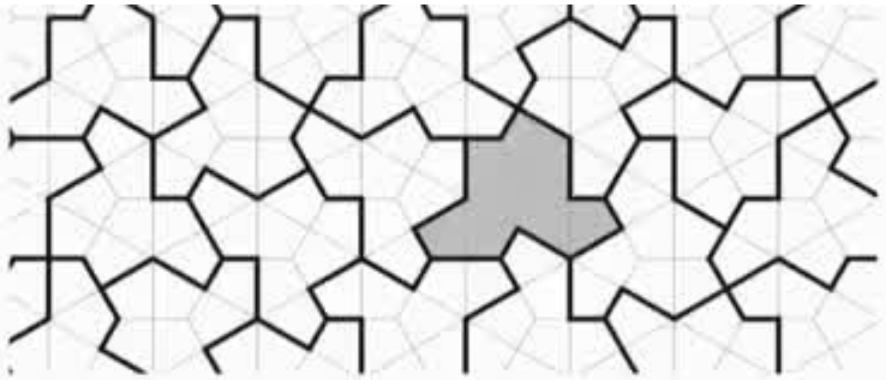


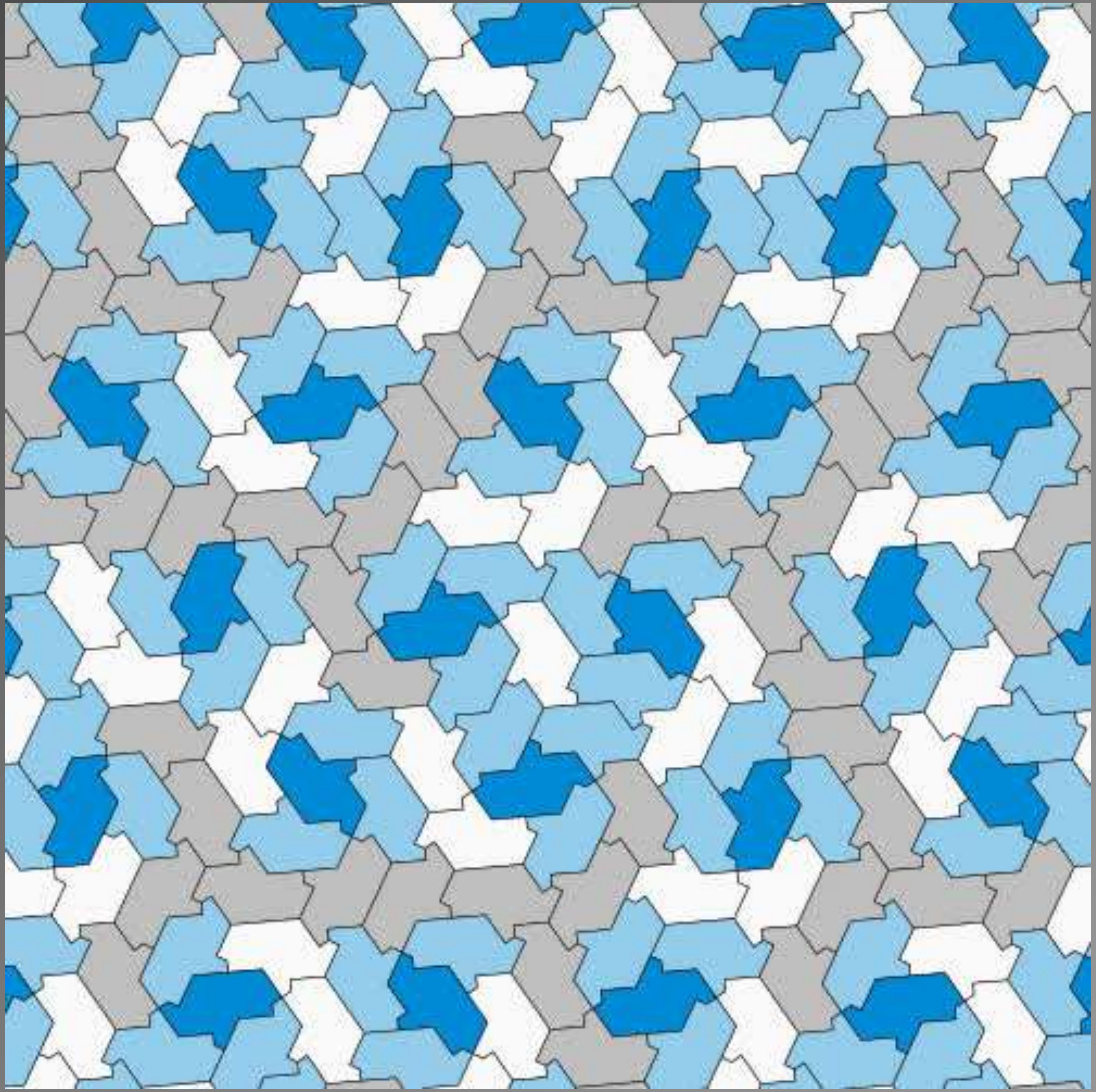
Figure 1.1: The grey "hat" polykite tile is an aperiodic monotile, also known as an "einstein". Copies of this tile may be assembled into tilings of the plane (the tile "admits" tilings), but none of those tilings can have translational symmetry. In fact, the hat admits uncountably many tilings. In Sections 2, 4, and 5 we describe how these tilings all arise from substitution rules, and thus all have the same local structure.

work on the then remaining open cases of Hilbert's *Entscheidungsproblem* [Wan61]. Wang encoded logical fragments by what are now known as *Wang tiles*—congruent squares with coloured edges—to be tiled by translation only with colours matching on adjoining edges. He conjectured that every set of Wang tiles that admits a tiling (possibly using only a subset of the tiles) must also admit a periodic tiling, and showed that this would imply the decidability of the *tiling problem* (or *domino problem*): the question of whether a given set of Wang tiles admits any tilings at all. The algorithm would consist of enumerating, for each positive integer n, the finite set of all legal $n \times n$ blocks of tiles. If there is no tiling by the tiles, there must be some n for which no such block exists (by the Extension Theorem [GS16, Theorem 3.8.1], which ultimately depends on the compactness of spaces of patches), and we will eventually encounter the smallest such n. On the other hand, if there is a fundamental domain for a periodic tiling, we will eventually discover it in a block. If Wang's conjecture held and aperiodic sets of tiles did not exist, this algorithm would always terminate.

Berger [Ber66] then showed that it was undecidable whether a set of Wang tiles admits a tiling of the plane. He constructed the first aperiodic set of 20426 Wang tiles, which he used as a kind of scaffolding for encoding finite but unbounded runs of arbitrary computation.

Subsequent decades have spawned a rich literature on aperiodic tiling, touching many different mathematical and scientific settings; we do not attempt a broad survey here. Yet there remain remarkably few really distinct methods of proving aperiodicity in the plane, despite or due to the underlying undecidability of the tiling problem.

Berger's initial set comprised thousands of tiles, naturally prompting the question of how small a set of tiles could be while still forcing aperiodicity. Professional and amateur mathematicians produced successively smaller aperiodic sets, culminating in discoveries by Penrose [Pen78] and others of several consisting of just two tiles. Surveys of these sets appear in Chapters 10 and 11 of Grünbaum and Shephard [GS16] and in an account of the Trilobite and Cross tiles [GS99]. A recent table appears in the work of Greenfeld and Tao [GT23b], counting



TILING

Link with mathematics

Current mathematics:

- Axioms: we start with a small number of statements, assumed to be a priori true
- Proofs: what is an implication, an equivalence, ...
- Theorems, lemma, corollaries

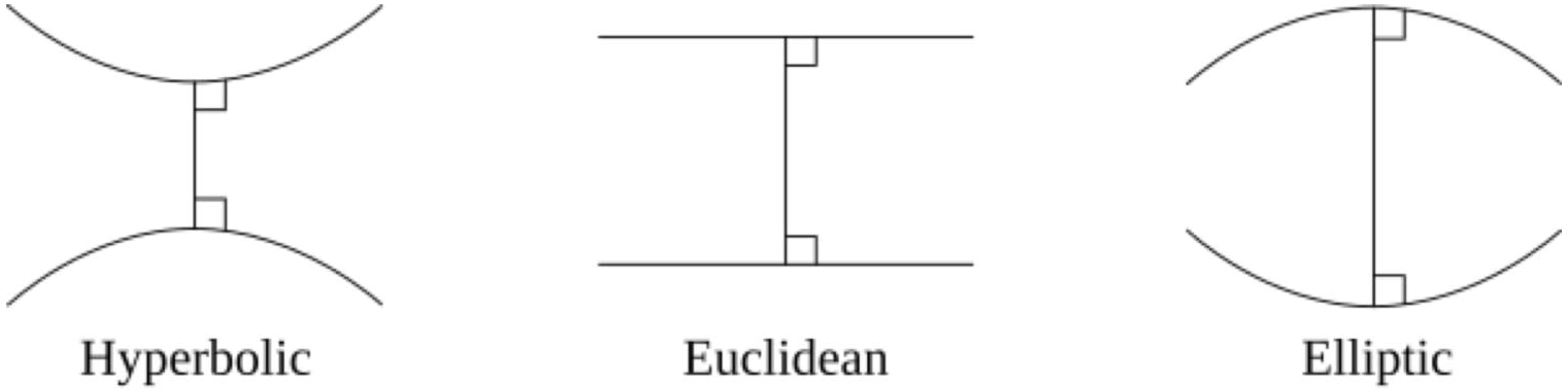
From the axioms, we therefore obtain theorems which gradually enrich mathematical theory. Because of the unproven bases (the axioms), the notion of "truth" of mathematics is subject to debate.

EXAMPLE OF AXIOMS USED FOR TILING

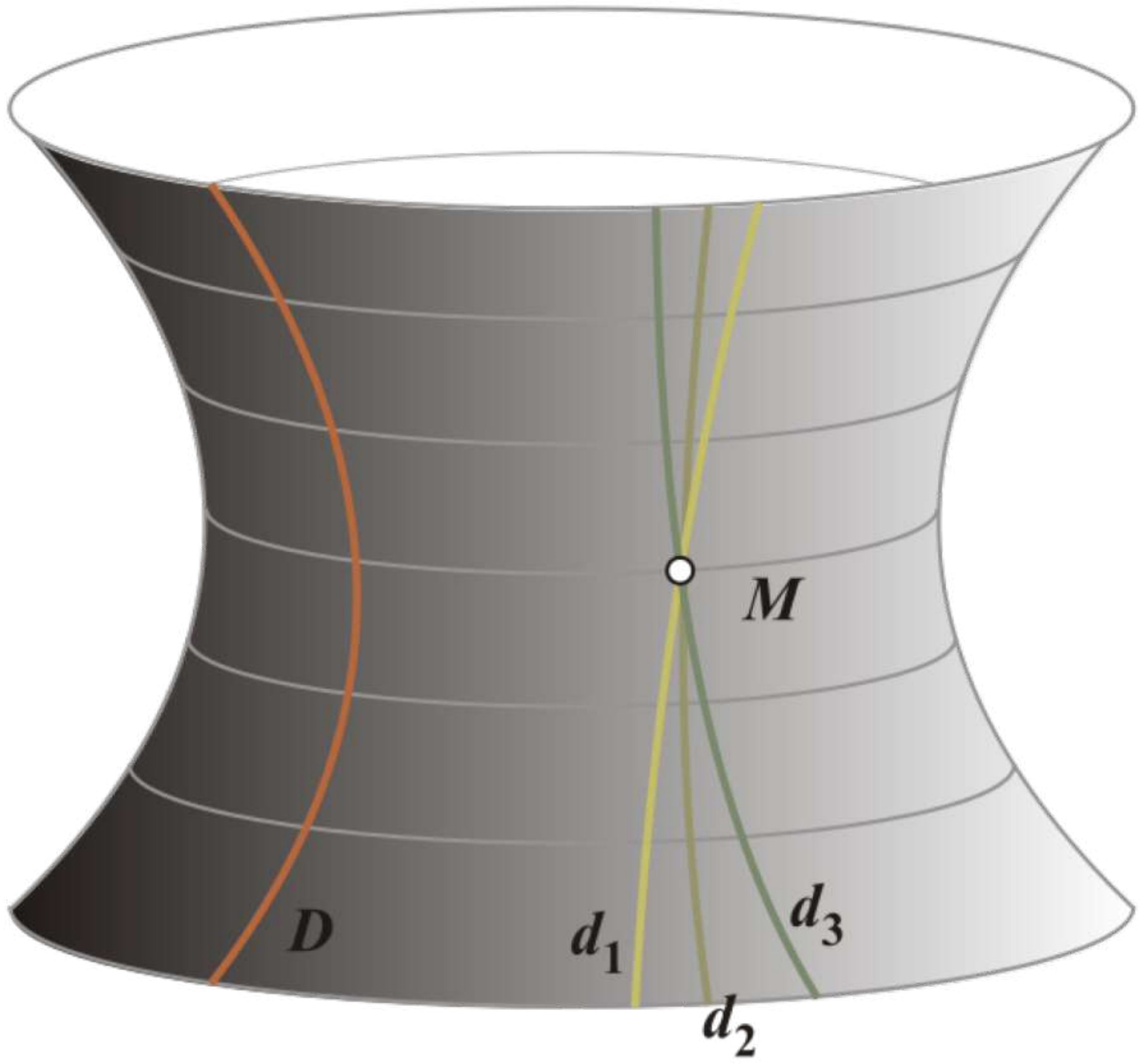
Euclid' axioms

[The parallel postulate] Given a line and a point not on it, at most one line parallel to the given line can be drawn through the point

What about other geometries?

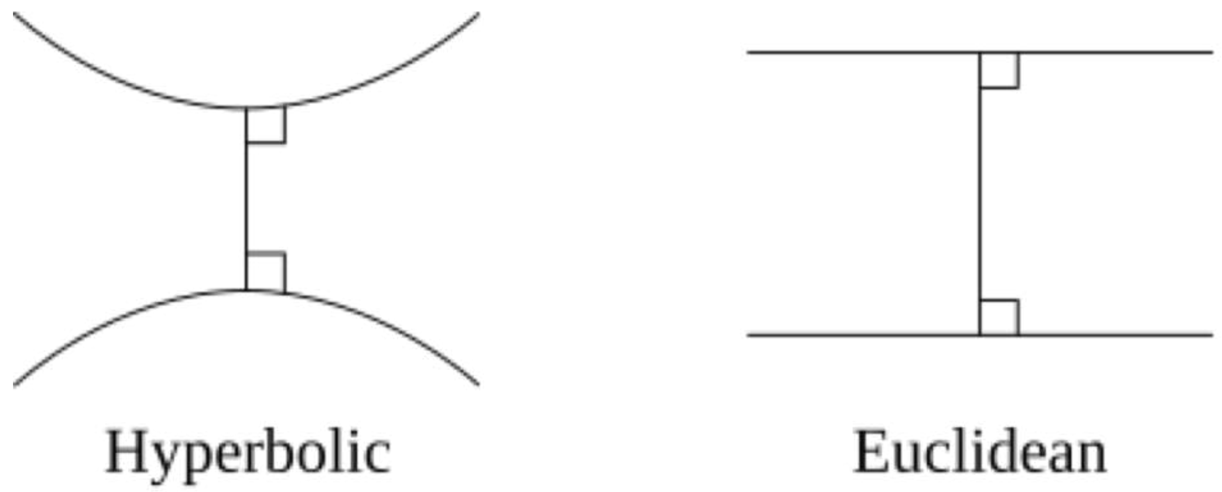


ELLIPTIC GEOMETRY

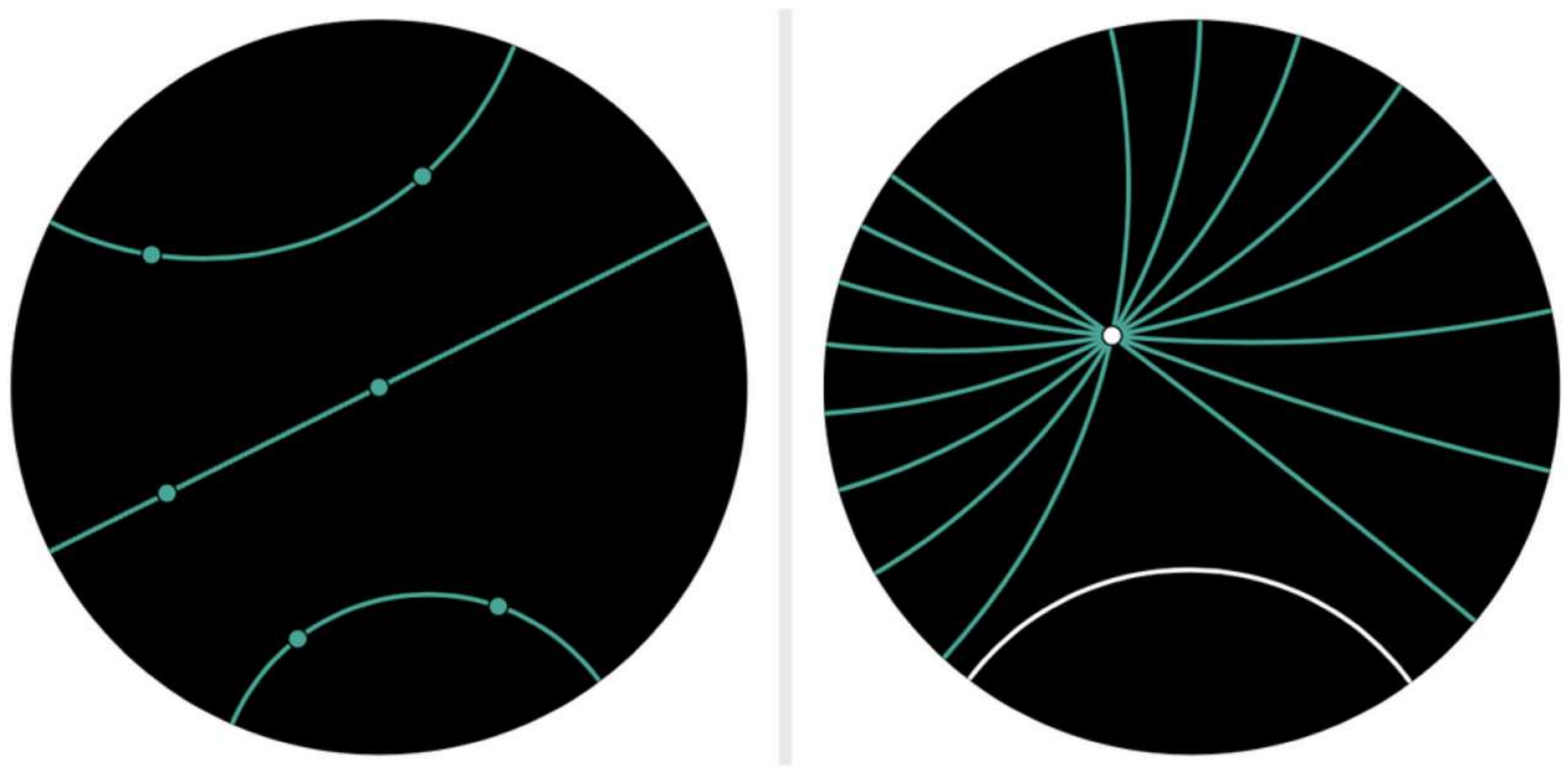


Hyperbolic geometry is a non-Euclidean geometry. The **parallel postulate** of Euclidean geometry is replaced with:

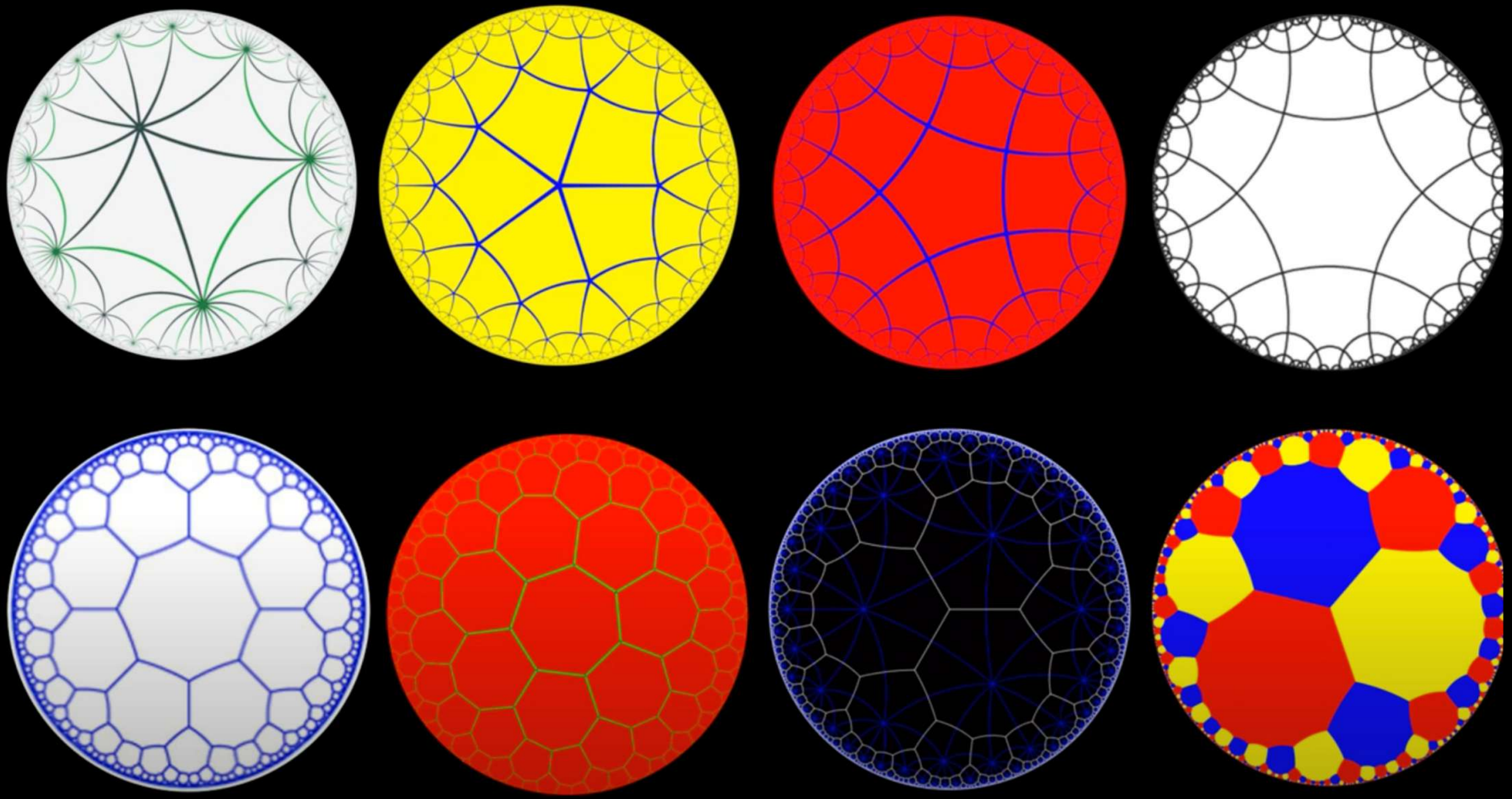
For any given line D and point M not on D, there are at least two distinct lines through M that do not intersect D.



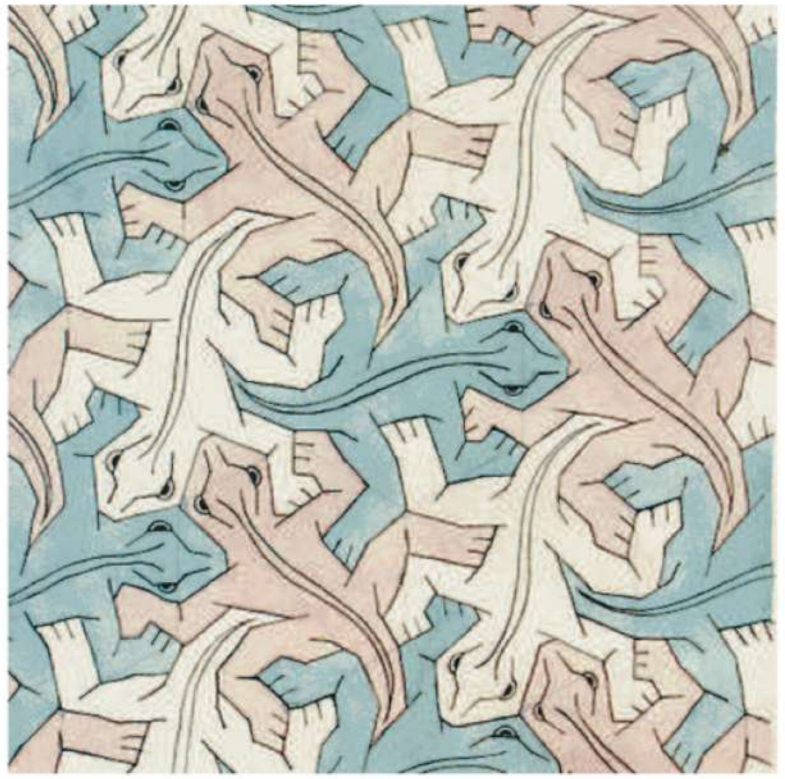
How can we tile the plan with hyperbolic geometry?

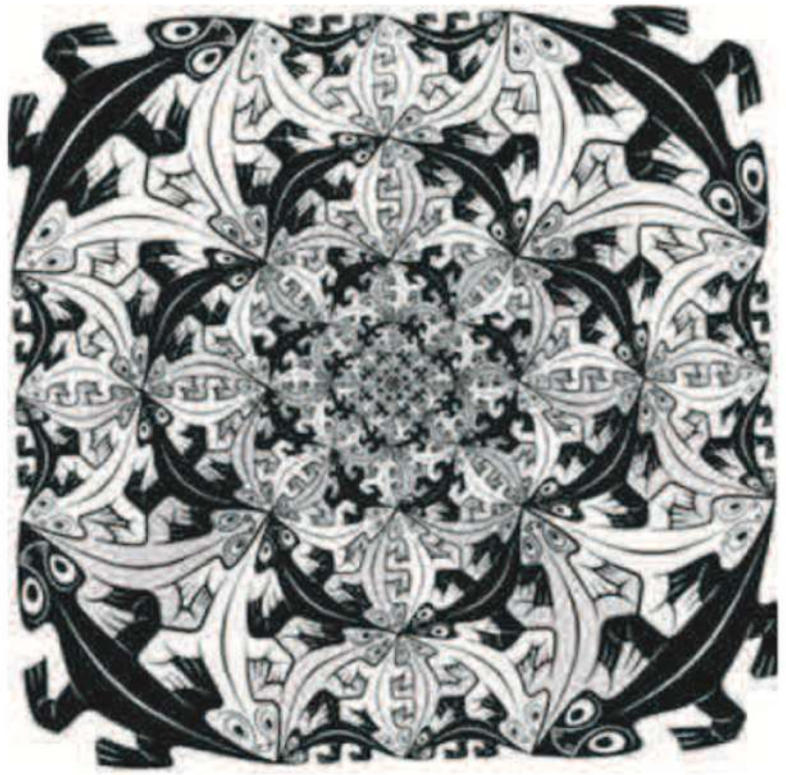


TILING IN HYPERBOLIC GEOMETRY



TILING IN HYPERBOLIC GEOMETRY



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Some websites to visit:

https://demonstrations.wolfram.com
https://tiled.art/en/home/
https://www.jaapsch.net/tilings/index.htm
Youtube: Thomaths